Chapter 2

INCENTIVE MECHANISMS

An incentive means motivating a subject to perform specific actions; in organizational systems, a principal stimulates an agent by exerting an impact on his or her preferences (i.e., the goal function).

The scientific research in the field of formal incentive models (within the framework of *control theory*) started in the late 1960s. The investigations were organized almost simultaneously and independently in the former USSR and in the USA, the UK, etc. The corresponding basic schools include the *theory of active systems* [9, 11, 12, 50], the *theory of hierarchical games* [18] and *contract theory* [8, 32, 57, 59]. Moreover, we underline that many incentive problems (demand for labor, labor offer, etc.) are traditionally studied in *labor economics* [42]. In addition, applied incentive problems are considered theoretically and widely used in the practice of *staff management*.

The present chapter focuses on the description of the basic approaches developed and the results obtained for the problems of incentive mechanism design. The material presented possesses the following structure. First, in Sections 2.1-2.2 we discuss incentive problems for a single-agent organizational system (in a continuous setting and in a discrete setting). Second, we describe the basic incentive mechanisms reflecting common forms and schemes of payment (Section 2.3). Next, Section 2.4 is devoted to incentive mechanisms in contract theory (notably, we treat an incentive problem for a single-agent system under conditions of stochastic uncertainty regarding the results of agent's activity). Finally, Sections 2.5–2.11 deal with different incentive mechanisms intended for a collective of agents (agents performing a joint activity).

2.1. INCENTIVE MECHANISM: A CONTINUOUS MODEL

In control theory, the major modeling tool for incentive problems consists in *game theory*, a branch of applied mathematics which analyzes models of decision-making in the conditions of noncoinciding interests of opponents (*players*); each player strives for the influence on the situation in his or her favor (see Appendix 1). An elementary game model is an interaction between two players–a superior (a *principal*) and a subordinate (an *agent*). Such organizational system has the following structure: the principal occupies the upper level of an hierarchy, while the agent is located at the lower level. For instance, a principal is an employer, an immediate superior of an agent, or an organization which has concluded an agreement with an agent (e.g., a labor contract, an insurance contract, a works contract, and so on). On the other hand, a wage worker, a subordinate employee, or an organization (being the second part in a corresponding contract) acts as an agent.

An agent's strategy is choosing an *action* $y \in A$ from a set of feasible actions A. In practice, an action means hours worked, product units manufactured, etc. The set of feasible actions represents a set of alternatives to-be-used by an agent in his or her choice (e.g., a range of possible working time, a nonnegative production output satisfying certain technological constraints, to name a few).

Let us introduce a series of definitions. An *incentive mechanism* is a principal's decision-making rule concerning rewards given to an agent. An incentive mechanism includes an incentive scheme; within the scope of models considered in this book, an incentive scheme is completely defined by its incentive function. By-turn, an incentive function specifies the relationship between the agent's reward (given by a principal) and the actions chosen by an agent. Therefore, in the sequel (to study game-theoretic models) we employ the terms "incentive mechanism," "incentive scheme" and "incentive function" as equivalents.

Given a set of feasible strategies M, a principal's strategy is choosing an *incentive function* $\sigma(\cdot) \in M$, mapping each action of an agent into a nonnegative reward paid to the agent, i.e., $\sigma: A \to \Re_1^+$. Constraints on the set of feasible rewards can be initiated by various legal acts (e.g., minimum wage) or can be adopted on the basis of economic expedience (of the principal's activity). Wage-rate schedules may be also used for an agent, and so forth.

By choosing the action $y \in A$, the agent incurs the *costs* $c(y)$; accordingly, the principal yields the *income* $H(y)$. We believe that the *agent's cost function* $c(y)$ and the *principal's income function* $H(y)$ are a priori known.

The interests of the organizational system participants (the principal and the agent) are determined by their *goal functions* (alternatively, by their payoff functions or utility functions–we omit dependence on the principal's strategy). Denote these functions by $\Phi(y)$ and $f(y)$, respectively.

The agent's goal function constitutes the difference between his or her reward and costs[1]:

$$f(y) = \sigma(y) - c(y). \tag{1}$$

At the same time, the principal's goal function represents the difference between his or her income and the *costs to motivate the agent*–the reward paid to the latter:

$$\Phi(y) = H(y) - \sigma(y). \tag{2}$$

Thus, we have defined the goal functions reflecting the preferences of the OS participants. Now, it seems reasonable to discuss distinction between financial incentives and non-financial recognition.

The presence of a scalar goal function implies the existence of a uniform equivalent used to measure all components of the goal function (the agent's costs, the principal's income, and, naturally, the reward itself).

[1] In this book, each section has independent numbering of formulas.

When we speak about financial incentives of an agent, the equivalent is money. Consequently, the principal's income has an obvious interpretation (furthermore, most of the research describing formal incentive models proceeds from expressing the agent's reward and the principal's income in money terms). A somewhat intricate matter concerns the agent's costs–measuring by money, e.g., the agent's satisfaction with his or her work is not always possible. In economic sense, one may understand the agent's costs as a money equivalent of his or her efforts required for choosing a specific action. Accordingly, it seems natural to suggest the idea of costs compensation; notably, the reward paid by the principal must (at least) cover the agent's costs. For a formal statement, see below.

Suppose the agent's costs are expressed in terms of some "utility" (e.g., bodily fatigue, "physic" income, and so on), and such utility has nothing in common with money. Moreover, the measuring units of the utility can not be reduced to the units of money by a linear transformation. In this case, summing up and subtracting the utility in the goal function (1) are well-posed only if we define the utility of a reward. For instance, financial incentive being involved, one may introduce the utility function $\tilde{u}(\sigma(y))$ that would describe the utility of money for an agent. The agent's goal function takes the form:

$$f(y) = \tilde{u}(\sigma(y)) - c(y).$$

Let us make the following assumptions to-be-used in this chapter (otherwise, special provisions are given).

First, suppose that the set of feasible actions of the agent is the positive real line; zero action means agent's non-participation in the OS (inaction).

Second, assume that the cost function is nondecreasing and continuous and vanishes in the origin (sometimes, we will also require its convexity and continuous differentiability).

Third, suppose that the income function is continuous, possesses nonnegative values, and attains the maximum under nonzero actions of the agent.

Fourth, assume that the reward paid to the agent by the principal is nonnegative.

We elucidate the assumptions below.

The first assumption implies that feasible actions of the agent are nonnegative real values, e.g., hours worked, production output, etc. Such structure of feasible actions determines "*continuous*" nature of the model considered in Section 2.1. The corresponding discrete model (with a finite set of feasible actions) of incentives is discussed in Section 2.2.

According to the second assumption, the choice of greater actions requires (at least) the same costs of the agent. For instance, the costs may grow following the rise in production output. In addition, zero action (agent's inactivity) causes no costs, while marginal costs[2] increase under greater actions; i.e., each subsequent increment of the action (by a fixed quantity) incurs higher costs.

The third assumption imposes certain constraints on the principal's income function by requiring that the principal benefits from the agent's activity. Evidently, no incentive problem exists otherwise (if the principal's income function attains the maximum under zero action of the agent, the former pays nothing to the latter and the agent is inactive).

The fourth assumption means that the principal does not penalize the agent.

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[2] In economics, marginal costs are defined as the derivative of a cost function.

The models of decision-making given in Chapter 1 state the following. A rational behavior of an OS participant lies in maximization of his or her goal function under all available information by choosing a proper strategy.

Let us define *awareness of players* and *sequence of moves*. Suppose that at the moment of decision-making (strategy choice) OS participants know all the goal functions and all the feasible sets. A specific feature of the game-theoretic incentive problem consists in a fixed sequence of moves (the game Γ_2 with side payments in the theory of hierarchical games, see Appendix 1 and [18]). A principal–a *metaplayer*–has the right to move first; thus, he or she reports the chosen incentive function to an agent. Under available information on the principal's strategy, the agent then chooses his or her action maximizing his or her goal function.

And so, we have described the basic parameters of an OS (a staff, a structure, feasible sets, goal functions, awareness and sequence of moves). Now, we pose the control problem proper–the problem of optimal incentive mechanism design.

Recall that the agent's goal function depends both on his or her strategy (action) and on the incentive function. According to the hypothesis of rational behavior, an agent chooses actions to maximize his or her goal function (for a given incentive function). Apparently, the set of such actions (referred to as the set of *implementable actions*) depends on the incentive scheme established by the principal. The central idea of motivation lies in that a principal stimulates agent's choice of specific actions by varying the incentive scheme.

Since the principal's goal function depends on the agent's action, the *efficiency of an incentive system* is the value of the principal's goal function on the set of the agent's actions implemented by the incentive scheme (in other words, on the set of actions chosen by the agent under the given incentive scheme). Consequently, the incentive problem is to find an optimal incentive scheme, i.e., an incentive scheme with maximal efficiency. We give formal definition below.

The set of agent's actions maximizing his or her goal function (depending on the incentive scheme adopted by a principal) is said to be a *solution set of the game* or a *set of implementable actions* (for the given incentive scheme):

$$P(\sigma) = Arg \max_{y \in A} \, [\sigma(y) - c(y)]. \tag{3}$$

Being aware that the agent chooses actions from the set (3), the principal must find an incentive scheme such that his or her goal function is maximized. Generally, the set $P(\sigma)$ is not a singleton; therefore, we have to redefine the agent's choice (in the sense of the principal's preferences regarding behavior of the agent). If special mention of the opposite takes no place, we believe that the *hypothesis of benevolence*[3] (HB) is valid. Then the agent chooses the most beneficial action for the principal from the set (3). A possible alternative for the principal is expecting the worst-case choice of the agent.

Consequently, the efficiency of the incentive scheme $\sigma \in M$ is defined by

[3] The hypothesis of benevolence consists in the following. If an agent is indifferent in choosing among several actions (e.g., actions ensuring the global maximum to his or her goal function), he or she definitely chooses the action being the most beneficial to a principal (the action maximizing the principal's goal function).

$$K(\sigma) = \max_{y \in P(\sigma)} \Phi(y), \qquad (4)$$

with the function $\Phi(y)$ being given by formula (2).

The hypothesis of benevolence being rejected, the guaranteed efficiency $K_g(\cdot)$ of the incentive scheme $\sigma \in M$ makes

$$K_g(\sigma) = \min_{y \in P(\sigma)} \Phi(y).$$

The direct incentive problem (the problem of optimal incentive scheme design) is to choose a feasible incentive scheme with maximal efficiency:

$$K(\sigma) \to \max_{\sigma \in M}. \qquad (5)$$

The inverse incentive problem is to find a set of incentive schemes implementing a given action or (in a general setting) a given set of actions $A^* \subseteq A$. For instance, in the case $A^* = \{y^*\}$, the inverse problem lies in finding the set of incentive schemes $M(y^*)$ that implement this action: $M(y^*) = \{\sigma \in M \mid y^* \in P(\sigma)\}$. The set $M(y^*)$ being evaluated, the principal can choose the "minimal" incentive scheme belonging to this set (in the sense of minimal costs to motivate the agent) or an incentive scheme with certain properties (e.g., monotonicity, linearity, etc.).

It should be emphasized that the above assumptions agree. An agent is always able to choose zero action causing no costs (the second assumption). At the same time, a principal can pay nothing to the agent for such action.

All interpretations of game-theoretic models of incentives presuppose a special alternative to an agent; notably, an agent may preserve the status-quo, i.e., not cooperate with a principal (concluding no labor contract with the latter). Not participating in a given OS, the agent does not obtain a reward from the principal; the former can always choose zero action and guarantee a nonnegative (more specifically, zero) value of the goal function. Suppose that outside a given OS an agent may receive a guaranteed utility $\overline{U} \geq 0$ (in contract theory, this is the *unemployment relief constraint* or the *reservation wage constraint*). Accordingly, in the case of participation in the OS considered the agent should be given (at least) the same level of utility. Taking into account the reservation utility, the set of implementable actions (3) takes the form

$$P(\sigma, \overline{U}) = Arg \max_{\{y \in A \mid \sigma(y) \geq c(y) + \overline{U}\}} \{\sigma(y) - c(y)\}. \qquad (6)$$

In the sequel, we assume zero reservation utility for simplicity of exposition.

Making a small digression, let us discuss the model of agent's decision-making in a greater detail. Suppose that a certain agent wants to get a job in an enterprise. He or she is suggested a contract $\{\sigma(y), y^*\}$, with a specific relationship $\sigma(\cdot)$ between the reward and the results y of his or her activity. In addition, the agent is informed of the expected results y^* of the activity. What are the conditions when the agent signs the contract? Note that both sides, *viz.*, the agent and the enterprise (the principal), make decision regarding signing of the contract independently and voluntarily. To answer, we start with consideration of some principles used by the agent.

The first condition–the *incentive compatibility constraint*–consists in the following. In the case of participation in a contract, the choice of the action y^* (only and exactly this action) maximizes the agent's goal function (utility function). In other words, the incentive scheme agrees with the interests and preferences of the agent.

The second condition–the *contract participation condition* (also known as the *individual rationality constraint*) claims the following. Signing a given contract, the agent expects gaining a greater utility than in the case of concluding an agreement with another organization (another principal). The agent's beliefs regarding possible income at the labor market are characterized by *reservation wage*; accordingly, the reservation wage constraint represents a special case of the individual rationality constraint.

Similar constraints of incentive compatibility and individual rationality can be applied to a principal, as well. Imagine there exists a single agent–a contender to sign a contract; then the contract will be beneficial to the principal under two conditions.

The first condition is analogous to the incentive compatibility constraint; it reflects conformity of the incentive scheme with the interests and preferences of the principal. Applying exactly the incentive scheme mentioned in the contract attains the maximum to the principal's goal function (utility function), see formula (4).

The second condition for the principal is identical to the contract participation constraint used for the agent. Notably, signing a contract with the agent appears more beneficial to the principal than preserving the status quo (rejection of the contract). For instance, assume that the profits of an enterprise (the principal's goal function) are zero without the contract. Then the profits must be strictly positive in the case of signing the contract.

Thus, we have discussed the prerequisites of mutually beneficial labor contracts between agents and principals. Getting down to formal analysis, we now solve the incentive problem (5). Note that a "frontal attack" in solving this problem seems rather difficult. Fortunately, one may guess the optimal incentive scheme based on certain considerations; after that, its optimality can be proved rigorously.

Assume that a certain incentive scheme $\sigma(\cdot)$ has been used by the principal, and under this scheme the agent has chosen the action $x \in P(\tilde{\sigma}(\cdot))$. Let another incentive scheme $\tilde{\sigma}(\cdot)$ be involved such that it vanishes everywhere with the exception of the *plan* (the point *x*) and coincides with the previous incentive scheme at the point *x*:

$$\tilde{\sigma}(y) = \begin{cases} \sigma(x), & y = x \\ 0, & y \neq x. \end{cases}$$

Then under the new incentive scheme the same action of the agent would ensure the maximum of his or her goal function.

Let us provide a formal proof of this assertion. The condition that the chosen action *x* ensures the maximum to the agent's goal function (provided that the incentive scheme $\sigma(\cdot)$ is used) could be rewritten in the following form. The difference between compensation and costs is not smaller than in the case of choosing any other action:

$$\sigma(x) - c(x) \geq \sigma(y) - c(y) \quad \forall y \in A.$$

Now, replace the incentive scheme $\sigma(\cdot)$ for that of $\tilde{\sigma}(\cdot)$ and obtain the following. The incentive scheme $\tilde{\sigma}(\cdot)$ still equals the incentive scheme $\sigma(\cdot)$ at the point x. The right-hand side of the expression includes the incentive system $\tilde{\sigma}(\cdot)$ which vanishes if $y \neq x$:

$$\sigma(x) - c(x) \geq 0 - c(y) \quad \forall y \neq x.$$

The first system of inequalities being valid implies the same for the second one. Hence, $x \in P(\tilde{\sigma}(\cdot))$ and the proof is completed.

Apparently, under the introduced assumptions the agent obtains (at least) zero utility by participating in the organizational system. The condition of agent's nonnegative utility

$$\forall\, y \in P(\sigma){:}\; f(y) \geq 0 \qquad (7)$$

forms the individual rationality constraint. Hence, (at the minimum) the actions are implementable such that the agent's goal function possesses nonnegative values (see formula (6)):

$$P_0(\sigma) = \{y \in A \mid \sigma(y) \geq c(y)\} \supseteq P(\sigma). \qquad (8)$$

Figure 2.1 shows the curves of the functions $H(y)$ and $(c\,(y) + \overline{U})$. According to the principal's viewpoint, a reward can not exceed the income gained by him or her from the agent's activity (indeed, the principal always obtains zero utility by refusing to cooperate with the agent). Hence, a feasible solution lies below the curve of the function $H(y)$. On the other hand, the agent believes a reward can not be smaller than the sum of the costs and reservation utility (zero action of the agent always leads to reservation utility). Consequently, a feasible solution lies above the curve of the function $c\,(y)$.

In addition, Figure 2.1 demonstrates the domain of actions being implementable based on the constraints of individual rationality ($\sigma(y^*) \geq c(y^*) + \overline{U}$) and incentive compatibility ($\forall\, y \in A$: $\sigma(y^*) - c(y^*) \geq \sigma(y) - c(y)$), taking into account non-negativity of the goal function.

The set of agent's actions and corresponding values of the goal function satisfying the above-mentioned constraints (incentive compatibility, individual rationality and others–both for the principal and for the agent) is said to be a "domain of compromise," see the shaded area in Figure 2.1. The set of agent's actions ensuring nonempty domain of compromise is given by

$$S = \{x \in A \mid H(x) - c(x) - \overline{U} \geq 0\}. \qquad (9)$$

Obviously, under fixed income and cost functions, increasing the quantity $\overline{U}$ gradually makes the domain of compromise degenerate.

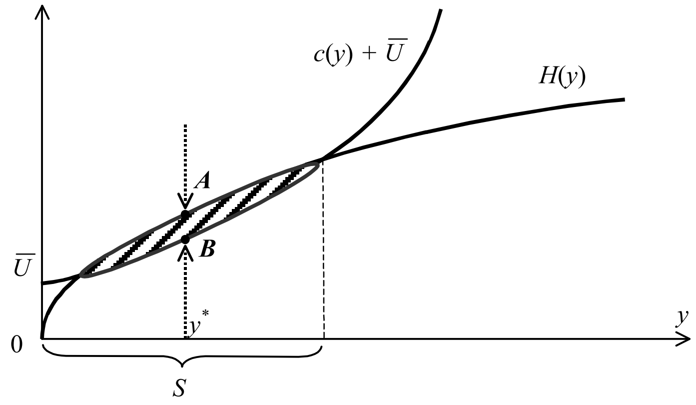

Figure 2.1. Optimal solution to the incentive problem.

Remember a principal seeks to minimize payments to an agent provided that the latter chooses the required action. This means that the point of optimal contract (under the hypothesis of benevolence) should be located on the lower boundary of the shaded domain in Figure 2.1 (the domain of compromise). In other words, *the* reward *should be exactly equal to the sum of the agent's costs and reservation utility.* This important conclusion is referred to as "*the principle of costs compensation.*" It claims that the principal has to compensate merely the agent's costs for motivating the agent to choose a specific action.

The principal may also establish a *bonus*[4] $\delta \geq 0$ (besides costs compensation). Therefore, for agent's choosing the action $x \in A$, the principal's incentive must be not smaller than

$$\sigma(x) = c(x) + \overline{U} + \delta. \tag{10}$$

We pay the reader's attention to the following. If an agent chooses certain actions (differing from the plan x) and his or her reward constitutes zero, then the constraints of incentive compatibility and individual rationality are satisfied for the agent. And the reward (10) given by the principal is maximal. Well, we have proven that a parametric solution to the problem (5) is defined by the incentive scheme:

$$\sigma_K(x, y) = \begin{cases} c(x) + \overline{U} + \delta, & y = x \\ 0, & y \neq x \end{cases}. \tag{11}$$

Here the parameter is $x \in S$. Such incentive schemes are referred to as *compensatory* (C-*type*) ones.

The principle of costs compensation forms a sufficient condition of implementability of the required action.

[4] Suppose that the hypothesis of benevolence takes no place; to find the most efficient incentive, the principal applies the principle of maximal guaranteed result on the maxima set of the agent's goal function. Formally, the bonus must be then strictly positive (but not arbitrarily small!). The hypothesis of benevolence remaining valid, the bonus may be zero. Generally, a bonus reflects the aspects of non-financial recognition.

Now, let us analyze what action should be implemented by the principal, i.e., what value $x \in S$ appears optimal.

According (10)–(11), the reward equals the agent's costs; thus, the optimal implementable action y^* maximizes (on the set *S*) the difference between the principal's income and the agent's costs. Hence, optimal implementable action is a solution to the following standard optimization problem:

$$y^* = arg\ \max_{x \in S}\ \{H(x) - c(x)\}, \qquad (12)$$

known as the *problem of optimal incentive-compatible planning* [9, 12]. Actually, the action to-be-performed by the agent (as the result of the principal's incentive) can be viewed as a *plan*–an action of the agent desired by the principal. The principle of costs compensation implies that the plan is *incentive-compatible* (recall an incentive compatible plan appears beneficial to the agent), and the principal has to find an incentive-compatible plan due to (11).

Within the framework of the hypothesis of benevolence, the value of the principal's goal function under optimal compensatory incentive scheme constitutes

$$\Delta = \max_{x \in S}\ \{H(x) - c(x)\}.$$

Assume that the income and cost functions are differentiable, the principal's income function is concave, while the agent's cost function is convex. In the model considered, the optimality condition for the plan y^* takes the form $\frac{dH(y^*)}{dy} = \frac{dc(y^*)}{dy}$. In economics, the quantity $\frac{dH(y)}{dy}$ is said to be the *marginal rate of production* (*MRP*), and $\frac{dc(y)}{dy}$ is the *marginal costs* (*MC*). The optimum condition (*MRP* = *MC*) determines the action y^* and corresponds to the so-called *effective wage* $c(y^*) + \overline{U}$.

Let us point out an important interpretation of the condition (12). The optimal plan y^* maximizes the difference between the principal's income and the agent's costs, i.e., attains the maximum to the sum of the goal functions (1)-(2) of the OS participants. Consequently, it turns out *efficient* in the sense of Pareto.

Note that the compensatory incentive scheme (11) is not the only optimal incentive scheme. One may easily show that, under the hypothesis of benevolence, a solution to the problem (5) is any incentive scheme $\breve{\sigma}(\cdot)$ meeting the condition $\breve{\sigma}(y^*) = c(y^*) + \overline{U}$, $\forall y \neq y^*$: $\breve{\sigma}(y) \leq c(y)$. See Figure 2.2, where we show three optimal incentive schemes–σ_1^*, σ_2^*, and σ_3^*.

The notion of a domain of compromise is extremely important methodologically. A nonempty domain of compromise implies a possibility of coordinating the interests of the principal and agent under the existing conditions. We clarify this.

In the formal model of incentives, strategies of the participants are limited by the corresponding feasible sets.

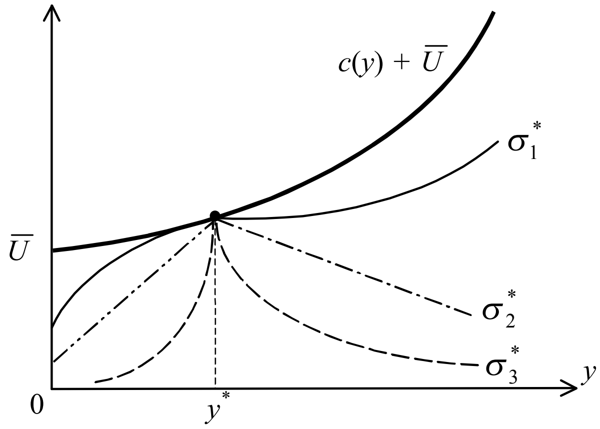

Figure 2.2. Optimal incentive schemes.

Rigorous consideration of the individual rationality constraint of the agent (we believe that the reservation wage parameter $\overline{U}$ reflects existing restrictions at the labor market) and of the principal (we believe that the nonnegative goal function of the principal reflects financial efficiency restrictions of the principal's activity, i.e., the costs to motivate the agent must not exceed the income gained by his or her activity), as well as of other incentive-compatibility conditions makes narrow the set of "rational" strategies–the domain of compromise.

In fact, a compromise between the principal and the agent consists in some utility allocation (actually, they divide the difference between the utilities in the points *A* and *B*, see Figure 2.1). By making the first move (i.e., offering a contract), the principal "appropriates" this difference, compelling the agent to agree with the reservation utility.

Consider the opposite situation when the agent makes the first move and suggests his or her contract to the principal; evidently, zero utility is obtained by the latter, while the former "collects" the difference between the utilities in the points *A* and *B*. Intermediate situations are also possible with an a priori fixed rule of allocating the income *AB* between the participants (based on a *compromise mechanism* [39, 50]).

The performed analysis implies that the incentive problem is solved in two stages. Notably, the *first stage* serves for treating the *coordination problem* (i.e., that of defining the set of implementable actions under given constraints). Next, at the *second stage* one deals with the *problem of optimal incentive-compatible planning* (searching for an implementable action being the most beneficial from the principal's viewpoint). Such decomposition approach is widely used in many sophisticated control problems for OS.

An essential advantage of compensatory incentive schemes consists in their simplicity and a high level of efficiency. Yet, an appreciable drawback is absolute instability against possible uncertainties of the model parameters. Indeed, suppose that the principal does not exactly know the agent's cost function. Then arbitrarily small inaccuracy may lead to considerable variations of actions implemented. The issues of incentive models adequacy and optimal solutions stability are studied in detail in [18, 50, 51]. The analysis framework and the methods of improving the guaranteed efficiency of incentives (proposed in the above-mentioned works under information being available to the principal) can be directly applied to the models considered below. Thus, here we skip over the issues of *adequacy* and *stability*.

Recall the optimal solution to the incentive problem derived above (i.e., when the principal adopts a compensatory incentive scheme). In the case of plan fulfillment, the value of the agent's goal function is equal to zero (or to the sum of the reservation utility and a bonus). Therefore, special attention (due to its wide spread occurrence) should be paid to the following situation. A labor contract (alternatively, an agreement between a customer (a principal) and an executor (an agent)) fixes the agent's profitability norm $\rho \geq 0$; in other words, the agent's reward depends on his or her action:

$$\sigma_\rho(x, y) = \begin{cases} (1+\rho)\, c(x), & y = x \\ 0, & y \neq x \end{cases}.$$

This is an *incentive scheme with profitability norm* [39, 50].

By assuming zero reservation utility of the executor, one obtains that the problem of optimal incentive-compatible planning takes the following form (compare with formula (12)):

$$y^*(\rho) = arg \max_{y \in A} \{H(y) - (1 + \rho)\, c(y)\}.$$

Hence, the maximum value of the principal's goal function constitutes:

$$\Delta(\rho) = H(y^*(\rho)) - (1 + \rho)\, c\,(y^*(\rho)).$$

Obviously, $\forall\ \rho \geq 0$: $\Delta(\rho) \leq \Delta$.

Consider an example. Set $H(y) = y$, $c(y) = y^2 / 2\,r$. Then we have: $y^*(\rho) = r / (1 + \rho)$, $\Delta(\rho) = r / 2\,(1 + \rho)$. The conditions of individual rationality imply that $\rho \geq 0$. In the example under consideration, the agent's income $\rho\, c\,(y^*(\rho))$ attains the maximum under $\rho = 1$, i.e., the agent benefits twice more from overstating the amount of work performed. According to the principal's viewpoint, zero profitability is preferable.

Thus, we have described the approach for studying the incentive problem based on the analysis of properties being intrinsic to the sets of implementable actions. Yet, there exists an alternative approach to investigate incentive problems. Above we have defined the set of actions implemented within a certain incentive scheme; after that, we have evaluated the maximum of the goal function (on this set) and chosen the corresponding incentive scheme. Note that solution process of the incentive problem has been decomposed into two stages, *viz.*, the *stage of interests' coordination* and the *stage of incentive-compatible planning*. This procedure possesses the following explicit representation. At the first stage, for each feasible incentive scheme $\sigma \in M$ one evaluates the set of implementable actions $P(\sigma)$ and "sums" them up: $P_M = \bigcup_{\sigma \in M} P(\sigma)$. At the second stage, one solves the problem of incentive-compatible planning, i.e., the maximization problem for the principal's goal function on the set P_M (also, see the general approach to control problems in organizations in Section 1.4).

Being able to solve the direct incentive problem, we easily find the solution to its inverse counterpart. For instance, the expression (9) allows for determining the minimal constraints to-be-imposed on rewards for implementability of given actions.

Interestingly, we have actually "guessed" the optimal solution without a "frontal attack" of the incentive problem[5]. The idea to introduce the sets of implementable actions has been of perceptible use here. An alternative technique lies in analyzing the minimal "costs" of the principal to motivate the agents[6]. Let us discuss it.

Suppose that the same action can be implemented within different incentive schemes. Apparently, greater efficiency is gained by an incentive scheme with smaller costs to motivate the agents. In other words, the optimal class of incentive schemes implements any action of the agent under the minimal principal's costs to motivate the latter. Despite its self-evidence, this statement provides a universal tool to solve incentive problems; below we will use it intensively.

Consider a class of feasible incentive schemes M; the minimal feasible reward that would stimulate agent's choosing a required action $y \in P_M$ is said to be *the minimal principal's costs to* implement the action. This quantity is defined by

$$\sigma_{\min}(y) = \min_{\sigma \in M} \{\sigma(y) \mid y \in P(\sigma), H(y) - \sigma(y) \geq 0\}. \qquad (13)$$

If actions are not implementable in the class M, the corresponding minimal costs to motivate implementation of these actions are assumed to be infinite:

$$\sigma_{min}(y) = +\infty, \; y \in A \setminus P_M. \qquad (14)$$

Hence, the introduced assumptions allow for reformulating the principle of costs compensation as $\forall \, y \in P_M : \sigma_{min}(y) = c(y)$.

We emphasize that the principle of costs compensation should not be understood *verbatim et literatim*. The agent's "costs" may include certain norms of profitability, and so on.

The notion of the minimal costs to motivate implementation of actions is of crucial importance. Analysis of these costs enables solving the design problem for optimal incentive function, studying the properties of optimal solutions, etc. [12].

Thus, we have shown that a C-type incentive scheme forms the optimal solution to the incentive problem (under the adopted assumptions). One would think, "What else can be "taken out" from this model?" The whole point is that we have supposed feasibility of compensatory incentive schemes. Unfortunately, in practice a principal may be restricted by a fixed class of incentive schemes. Such restrictions are subject to exogenous factors (e.g., legal norms for wages) or to endogenous factors (e.g., the principal inclines towards piece wage or time-based wage instead of simple compensation of costs–see Section 2.3).

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[5] We should admit the following aspect. In the theory of control in organizations, a researcher often guesses solutions (based on intuition, meaningful considerations, etc.) and strives for deriving the corresponding analytical solution. The reasons are clear enough; indeed, the analysis of a formal model of an organizational system is not an end in itself for a researcher. Quite the contrary, he or she has to propose the most adequate (reality-consistent and easily interpretable) solution to a control problem.

[6] Let us make a remark on terminology. The notion "costs" characterizes the costs of an agent to choose a specific action. On the other part, the notion "costs to motivate implementation of an action" characterizes the costs required for the principal to stimulate implementation of the action.

2.2. INCENTIVE MECHANISM: A DISCRETE MODEL

Let us formulate and solve the *discrete incentive problem* in a two-level OS which consists of a principal and a single agent. Actually, this is the discrete counterpart of the problem studied in Section 2.1 (an incentive problem is said to be discrete if the agent's set of feasible actions appears finite).

Solution to the discrete incentive problem. Assume that an agent has a finite set of feasible actions: $N = \{1, 2, ..., n\}$. In the absence of incentives, the agent's preferences form the vector $q = (q_1, q_2, \ldots, q_n)$, whose components mean the income gained by the choice of a corresponding action. The principal's control lies in choosing an incentive scheme $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, i.e., in paying a reward (negative or positive) to the agent for specific actions. We will believe that the rewards are unbounded. The agent's "goal" function $f = (f_1, f_2, \ldots, f_n)$ represents the sum of the income and the reward: $f_i = q_i + \sigma_i$, $i \in N$. Within the hypothesis of rational behavior (see Section 1.1) and under a known incentive scheme, the agent chooses an action maximizing his or her goal function. Imagine that several such actions exist; in this case, we suppose the agent chooses the action being the most beneficial to the principal (in a certain sense)–the hypothesis of benevolence takes place. The efficiency of an incentive scheme (a control mechanism) is the maximal value of the principal's goal function on the set of agent's action implementable under the incentive scheme.

The incentive problem consists in establishing a certain incentive scheme (by the principal) such that the agent chooses the most beneficial action to the principal. Solution to this problem seems elementary (see Section 2.1). Indeed, for a fixed incentive scheme σ, one defines the set of agent's actions attaining the maximum to the agent's goal function (referred to as the set of implementable actions): $P(\sigma) = \{i \in N \mid f_i \geq f_j, j \in N\}$. After that, one searches for an incentive scheme which implements the action being most beneficial to the principal.

Assume that the principal's goal function $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ makes up the difference between the income and the incentive, i.e., $\Phi_i = H_i - \sigma_i$, $i \in N$. Consequently, we obtain the following optimal incentive scheme:

$$\sigma^* = arg\ \max_{\sigma}\ \max_{i \in P(\sigma)} \{H_i - \sigma_i\}.$$

Rewrite the set of implementable actions as $P(\sigma) = \{i \in N \mid q_i + \sigma_i \geq q_j + \sigma_j, j \in N\}$. The *minimal incentive scheme* (i.e., possessing the minimal value at each point) which implements *all* actions of the agent under the hypothesis of benevolence, is the compensatory incentive scheme $\sigma^K = (\sigma_1^K, \sigma_2^K, \ldots, \sigma_n^K)$. It is determined by

$$\sigma_j^K = q_k - q_j, j \in N, \tag{1}$$

where $k = arg \max_{j \in N} q_j$.

The set of optimal implementable actions (according to the principal's view) takes the form

$$P(\Phi, f) = Arg \max_{i \in N} \{H_i - \sigma_i^K\} = Arg \max_{i \in N} \{H_i - q_k + q_i\}. \quad (2)$$

Being a solution to the incentive problem, the compensatory incentive scheme *in medias res* makes all feasible actions of the agent equivalent (in the sense of his or her goal function values). In other words, this scheme compensates exactly the costs incurred by the agent as the result of choosing the required action (as against choosing the action k which yields the maximum gain in the absence of incentives). Indeed, the principal should not pay excess for the choice of the action k.

Thus, if we state the incentive problem in terms of the agent's goal functions, his or her preferences on a finite set are defined by the vector q. Components of the vector are certain values, and the differences between them (see formula (1)) represent minimal payments that make the corresponding pairs of actions equivalent (in the sense of values of the agent's goal function values). An alternative approach is to describe the preferences directly on the pairs of agent's actions. In other words, one can just enumerate $n^2 - n$ values (e.g., expert information obtained during paired comparison of different options); they mean relative preference of actions in the sense of minimal excess payments required for equivalence of the corresponding pair. Such technique and its correlation with the method of defining preferences in terms of the goal functions are discussed in the current section.

The incentive problem stated in terms of metrized binary relations. The agent's goal function depends on the incentive scheme established by the principal; accordingly, this function generates a complete antisymmetric transitive *binary relation* on the set N (see Appendix 3). Moreover, one may always find (at least, a single) alternative (an action) being *undominated* with respect to this relation. The incentive problem allows for the following statement in terms of the binary relation: find an incentive scheme such that the most beneficial alternative for the principal is undominated.

However, the above statement of the problem seems somewhat artificial. First, we loose a practically relevant interpretation of the incentives as the rewards for choosing specific actions (really, making the binary relation explicitly dependent on the incentive vector is rather exotic). Second, the same binary relation can be generated by different goal functions (not necessarily identical up to an additive constant–see Appendix 4). Finally, the way of performing the reverse transition (from a binary relation to a corresponding goal function) is not totally clear. In applications, exactly numerical values of the agent's rewards play the key role.

An intermediate position between "standard" binary relations and goal functions is occupied by the so-called *metrized relations* (MR). On the set N, a MR is specified by the matrix $\Delta = ||\delta_{ij}||$, $i, j \in N$. The elements δ_{ij} of the matrix Δ ($i, j \in N$) are positive, negative or zero values representing comparative preferences of different alternatives (in our case, these are actions of the agent). Note that we consider *complete relations*, i.e., incomparability of actions is impossible, and so on.

Suppose that if $\delta_{ij} <$ (>) 0, then action i is strictly better (worse, respectively) for the agent than action j (in the absence of incentives). Actions i and j are equivalent for the agent if $\delta_{ij} = 0$. In practice, δ_{ij} constitutes the sum to-be-paid excess to the agent for making action i equivalent to action j.

Next, assume that the principal's control (incentive) lies in modifying the comparative preference of different actions, i.e., elements of the matrix Δ. As usual, the incentive problem is to find a feasible variation of the elements such that the most beneficial action (according to the principal's view) coincides with the best action for the agent.

Let the agent's preferences satisfy the following property: $\forall\ i, j, m \in N$: $\delta_{im} + \delta_{mj} = \delta_{ij}$ (referred to as the *internal consistency condition* (ICC) of preferences). The ICC implies that $\delta_{ii} = 0$, $\delta_{ij} = -\delta_{ji}$, $i, j \in N$ (see the section devoted to pseudopotential graphs in Appendix 2). Moreover, the graph which corresponds to the matrix Δ is a potential one with q_i ($i \in N$) as the node potentials. The latter are evaluated up to an additive constant as follows:

$$q_i = -\frac{1}{n}\sum_{m=1}^{n}\delta_{im},\ i \in N. \tag{3}$$

The matrix Δ can be uniquely recovered using the potentials q_i, $i \in N$:

$$\delta_{ij} = q_j - q_i,\ i, j \in N. \tag{4}$$

The potentials of actions may be treated as values of the agent's income function, while elements of the matrix Δ serve as their first differences.

Imagine that the agent's preferences are specified in the form of a MR which satisfies the ICC. In this case, information on all elements of the matrix Δ turns out redundant. For instance, a certain row (or column) being known, the ICC enables easy recovery of the rest elements by summing over the corresponding chains. Such property of an internally consistent MR seems attractive in the aspect of the amount of information required for correct identification of OS parameters.

For the agent, the best action within the model considered consists in an action k such that $\delta_{kj} \le 0$ for all $j \in N$. In the case of internally consistent preferences, the described action (probably, not unique) always exists. Actually, this is the action possessing the maximal potential. Hence, the set of implementable actions forms $P(\Delta) = \{k \in N \mid \delta_{kj} \le 0, j \in N\}$.

Consider an arbitrary pair of actions, i and j ($i, j \in N$), and define the "equalizing" operation $(j \to i)$ for their potentials: $q_j^{j\to i} \to q_j + (q_i - q_j)$. In terms of elements of the matrix Δ, the operation includes two steps:

1) $\delta_{jm}^{j\to i} \to \delta_{jm} + \delta_{ij}$, $m \in N$;
2) $\delta_{mj}^{j\to i} \to -\delta_{jm}$, $m \in N$.

Obviously, action j becomes equivalent to action i ($\delta_{ij} = \delta_{ji} = 0$), and the internal consistency condition for the agent's preferences is preserved. For the principal, the costs to perform the operation $(j \to i)$ constitute $\delta_{ji} = q_i - q_j$ (see (1)).

The idea to solve the incentive problem lies in the following. To stimulate agent's choosing an action $l \in N$, the principal must pay the agent the reward σ_l meeting the system of inequalities $\sigma_l - \sigma_i \ge \delta_{li}$, $i, l \in N$. The compensatory incentive scheme

$$\sigma_l = \max_{j \in N} \delta_{lj} = \max_{j \in N} (q_j - q_l) = q_k - q_l = \delta_{lk},\ l \in N, \tag{5}$$

satisfies this system of inequalities. Consequently, if k is the most beneficial action for the agent (in the absence of incentives), then the minimal value of the incentive σ_l required for implementing the action l equals δ_{lk}, $l \in N$. Again, we emphasize that the compensatory incentive scheme (5) makes ***all*** actions of the agent equivalent in his or her opinion.

Suppose that the agent's preferences (in the absence of incentives) are given by a MR, i.e., by the matrix $\Gamma = ||\gamma_{ij}||$, $i, j \in N$, and the ICC takes place. One may set up a correspondence between the matrix Γ and the principal's income "function" $H_i = -\frac{1}{n}\sum_{m=1}^{n}\gamma_{im}$, $i \in N$. Assume that the reward paid to the agent is deduced from the principal's income function; implementing the action l, the principal then "looses" δ_{lk}, $l \in N$. Therefore, the comparative preference of the actions' pair (k, l) changes in the eyes of the principal, as well. Due to the ICC, the new value is the sum $(\gamma_{kl} + \delta_{kl})$. Hence, the principal's preferences (taking into account the incentives) are represented by the metrized relation $\Xi = \Delta + \Gamma = ||\gamma_{ij} + \delta_{ij}||$, $i, j \in N$.

Note that the principal's preference relation Ξ additively includes his or her own preferences and the agent's preferences (both preferences are in the absence of incentives). This fact allows for interpreting the incentive as coordination of their interests.

Apparently, if the preferences of the agent and of the principal are internally consistent (in the absence of incentives), then the metrized relation Ξ meets the ICC. This leads to the following assertion [50]: the set of optimal implementable actions of the agent is given by (compare with formula (2))

$$P(\Gamma, \Delta) = \{i \in N \mid \delta_{ij} \le \gamma_{ji}, j \in N\}.$$

The correlation of the incentive problems formulated in terms of goal functions and MR is represented by the following statement [50]: the incentive problems formulated in terms of goal functions and MR that satisfy the ICC are equivalent.

The equivalence means mutual reducibility of these problems. Suppose that the incentive problem is stated in terms of goal functions, i.e., we know the agent's income function q. We believe that the values of the income function are potentials and define the matrix Δ by formula (4). One would easily see that the ICC is valid. Similarly, under the ICC, the matrix Δ can serve for restoring the potentials (the income function) by means of the expression (3). Thus, we perform the reverse transition. And so, under the ICC formulas (3)–(4) imply that $P(\Gamma, \Delta) = P(\Phi, f)$.

The conducted analysis shows that MR describe a wider class of the preferences (both for the agent and the principal) than goal functions. In fact, the latter are equivalent to internally consistent MR.

Of course, one has no guarantee that an MR (obtained in practice, e.g., by an expertise and reflecting the preferences of a controlled subject) appears internally consistent. The methods to solve the incentive problems stated in terms of MR without the ICC are described in [50].

2.3. BASIC INCENTIVE MECHANISMS

Let us discuss *the basic incentive schemes* (*mechanisms*) in single-agent deterministic organizational systems, i.e., the systems operating under complete information on all essential (internal and external) parameters. *A compensatory* (*C-type*) *incentive scheme* is the optimal basic incentive scheme, see Section 2.1.

A jump (J-type) *incentive scheme* is characterized by the following: the agent obtains a fixed reward C provided that his or her action appears not smaller than a planned action x; otherwise, the agent has zero reward (see Figure 2.3):

$$\sigma_J(x, y) = \begin{cases} C, y \geq x \\ 0, y < x \end{cases}. \tag{1}$$

The parameter $x \in X$ is said to be a *plan*, i.e., a state (an action, a result of activity, etc.) of an agent desired by a principal.

J-type incentive schemes may be treated as a *lump sum* payment which corresponds to the reward C under a given result (e.g., an output level being not smaller than a predetermined threshold, working hours, and so on). Another interpretation is when the agent is paid for hours worked; for instance, the reward then corresponds to a fixed wage under full-time occupation.

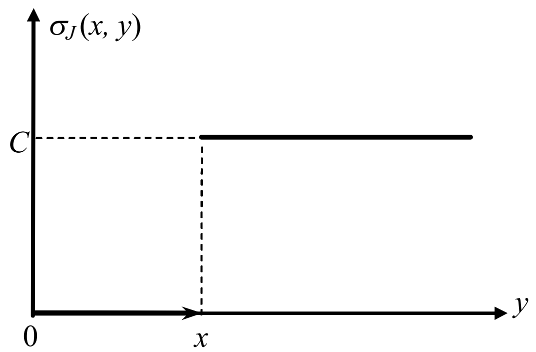

Figure 2.3. A jump incentive scheme.

Proportional (linear) (L-type) incentive schemes. Fixed reward rates are widely used in practice. For instance, a fixed per-hour rate implies the same *wage is paid* for every hour worked, while piece-based wage implies a fixed reward for every unit of the manufactured product. In both schemes the agent's reward is proportional to his action (hours worked, product units manufactured, etc) and the wage rate $\alpha \geq 0$ represents the proportionality coefficient (see Figure 2.4):

$$\sigma_L(y) = \alpha\, y. \tag{2}$$

In the general case, some reward is given to an agent regardless of his or her actions, i.e., proportional incentive schemes take the form:

$\sigma_L(y) = \sigma_0 + \alpha\, y.$

Suppose that a linear incentive scheme is employed and the cost function of the agent is continuously differentiable, monotonous and convex. Optimal action y^* of the agent (maximizing his or her goal function) is defined by the formula $y^* = c'^{-1}(\alpha)$, where $c'^{-1}(\cdot)$ stands for the inverse derivative of the agent's cost function. Note that the principal more than compensates the agent's cost by choosing the action y^*; actually, the principal overpays the following amount: $y^* c'(y^*) - c(y^*)$. For instance, suppose the agent has the income function $H(y) = b\, y,\ b > 0,$ while his cost function is convex: $c(y) = a\, y^2,\ a > 0$. In this case, for any feasible action of the agent the principal pays twice compared to the optimal payment.

Therefore, under a convex cost function of the agent, efficiency of the proportional scheme is not greater than that of the compensatory one. A curve of the agent's goal function (under a proportional incentive scheme used by the principal) is demonstrated by Figure 2.5.

Low efficiency of proportional incentive schemes described by the formula $\sigma_L(y) = \alpha y$ is subject to non-negativity of rewards. Assume that the reward may be negative for some actions (note that these actions are probably never chosen, as shown in Figure 2.6), i.e., $\breve{\sigma}_L(y) = \sigma_0 + \alpha\, y$, with $\sigma_0 \le 0$. Then, under a convex cost function of the agent, efficiency of the proportional incentive scheme $\breve{\sigma}_L(\cdot)$ could equal that of the optimal (compensatory) incentive scheme.

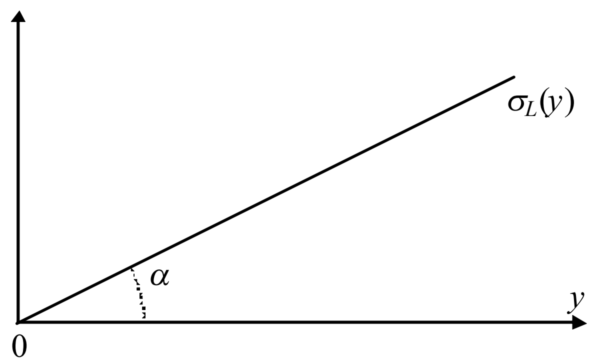

Figure 2.4. A proportional incentive scheme.

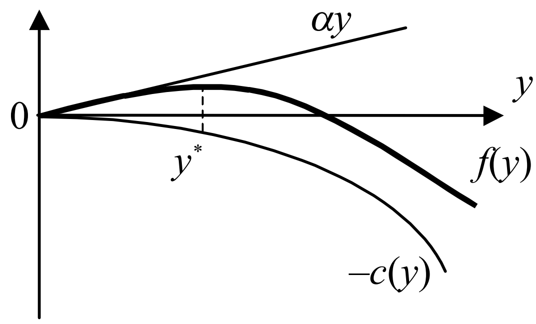

Figure 2.5. A goal function of the agent: the principal uses an L-type incentive scheme.

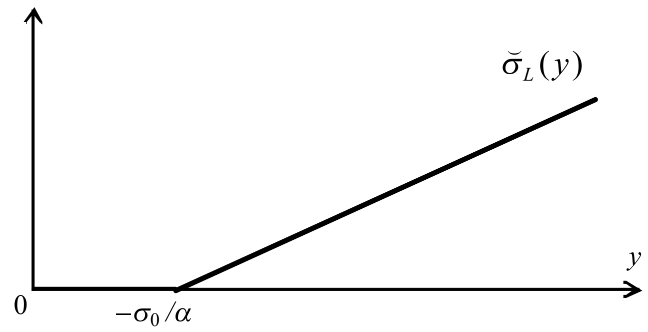

Figure 2.6. A linear incentive scheme.

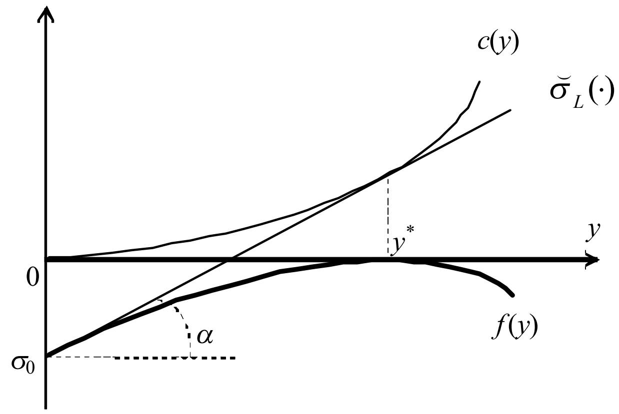

Figure 2.7. A goal function of the agent: the principal uses the incentive scheme $\breve{\sigma}_L(\cdot)$.

It suffices to involve the following expressions to substantiate the above assertion (see Figure 2.7, as well):

$$y^*(\alpha) = c'^{-1}(\alpha),\ \sigma_0(\alpha) = c(c'^{-1}(\alpha)) - \alpha\, c'^{-1}(\alpha).$$

The optimal value α^* of the wage rate is chosen from the maximum condition for the goal function of the principal:

$$\alpha^* = arg \max_{\alpha \geq 0} \ [H(y^*(\alpha)) - \breve{\sigma}_L(y^*(\alpha))].$$

Incentive schemes based on income redistribution (*D-type*) employ the following idea. Since the principal represents preferences of the whole system, principal's income can be equated to that of the whole organizational system. Hence, one may base an incentive scheme of the agent on the income obtained by the principal; in other words, one may set the agent's reward equal to a certain (e.g., fixed) share $\xi \in [0; 1]$ of the principal's income:

$$\sigma_D(y) = \xi\, H(y). \tag{3}$$

We underline that *C*-, *J*-, *L*-, and *D*-type incentive schemes are parametric. Notably, it suffices to choose the pair (x, C) for specifying the jump incentive scheme. Defining the proportional incentive scheme requires designating the wage rate α. Finally, one should only select the income share ξ to describe the incentive scheme based on income distribution.

The incentive schemes described above are elementary and serve as blocks of a "kit"; using these blocks, it is possible to construct complex incentive schemes (referred to as *secondary schemes* with respect to the basic ones). Thus, we should define operations over the basic incentive schemes for making such a "construction" feasible. Dealing with a single-agent deterministic OS, the researcher may confine himself to the following three types of operations.

The first-type operation is the transition to a corresponding incentive *quasi-scheme*, i.e., the reward is assumed to be zero everywhere, with the exception of the planned action. In the complete information framework, "nulling" the incentive in all points (except the plan) does not modify the properties of the incentive scheme under the hypothesis of benevolence. Therefore, in the sequel we will not dwell on differences between a specific incentive scheme and its counterpart (a scheme derived from the initial scheme by means of the first-type operation).

The second-type operation is the composition, i.e., employing different basic incentive schemes in different subsets of the set of feasible actions. The resulting incentive systems are called *composite*.

The third-type operation is represented by algebraic addition of two incentive schemes (this is possible as the reward enters the goal functions additively). The result of such operation is referred to as a *cumulative incentive scheme*.

For instance, Figure 2.8 shows an incentive scheme of *J+L* type (a tariff plus-bonus incentive scheme), derived by summing-up jump and linear incentive schemes.

Thus, the *basic incentive schemes* include the ones of *J*-, *C*-, *L*-, and *D*- type, as well as any schemes derived from them through the above-mentioned operations.

First, it is shown in [50] that the incentive schemes derived for the basic incentive schemes discussed cover all personal wage systems used in practice. Second, the cited work provides some estimates of comparative efficiency for different combinations of the basic incentive schemes.

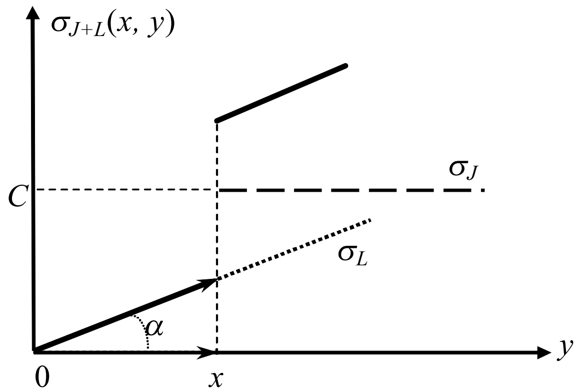

Figure 2.8. An incentive scheme of J+L type (a cumulative incentive scheme).

2.4. INCENTIVE MECHANISMS IN CONTRACT THEORY

Contract theory is a branch of control theory for socio-economic systems, studying game-theoretic models of interaction between a control subject (a *principal*) and a controlled subject (an *agent*) under the conditions of external stochastic uncertainty [8].

In models of contract theory, the uncertainty is taken into consideration by the following technique. The *result of agent's activity* $z \in A_0$ represents a random variable whose realization depends on the agent's actions $y \in A$ and on an external uncertain parameter, known as the *state of nature* $\theta \in \Omega$. A state of nature reflects external conditions of agent's activity that affect the result of activity, thus making it different from the action.

The participants of OS possess the following awareness. At the moment of decision-making, they know the probability distribution $p(\theta)$ of the state of nature or the conditional probability distribution $p(z, y)$ of the activity result. The principal does not observe actions of the agent; instead, the former merely learns the result of the latter's activity. At the moment of choosing his or her action, the agent may know either the state of nature (asymmetric awareness) or the corresponding probability distribution (symmetric awareness). It should be emphasized that the second case better fits the incentive models; consequently, we focus on it below.

The principal's strategy is choosing a certain function $\sigma(\cdot)$ according to the result of agent's activity. Depending on possible interpretations of the model, this function represents an incentive function (labor contracts), an insurance compensation (insurance contracts), debts or payments (debt contracts), and so on. The agent's strategy lies in choosing an action under a known strategy of the principal. A *contract* is a set of strategies of the principal and of the agent. Note there exist explicit contracts being legally valid (e.g., the majority of insurance contracts and debt contracts) and implicit contracts being tacit or not concluded *de facto* (e.g., labor contracts in some situations).

The result of agent's activity depends on uncertain parameters and defines the utilities of the OS participants. Therefore, we assume that (making their decisions) the participants average the utilities with respect to known probability distribution and choose the strategies by maximizing their expected utility.

An optimal contract is the most beneficial to the principal (as attaining the maximum to his or her goal function) provided that the agent benefits from interaction with the principal. From the agent's viewpoint, this means that the contract participation condition and the individual rationality condition must be satisfied (similarly to the model considered in Section 2.1).

The pioneering works on contract theory appeared in the early 1970s. That research involved game-theoretic models as an endeavor to explain the existing contradictions between the results of macroeconomic theories and the actual rates of unemployment and inflation in developed countries at that time.

Notably, one of the "contradictions" was the following. There are three types of wage, *viz.*, a *market wage* (the reservation utility being guaranteed to an employee), an *efficient wage* (the payments maximizing the employee's efficiency in the sense of the whole enterprise; as a rule, the efficient wage is defined by the equality between the marginal product yielded by the employee and his or her marginal costs) and an *actual wage* (the one given to the employee). Statistical data indicated that the actual wage differed from its efficient counterpart.

The first models of contract theory analyzed the problems of optimal number of employees under the participation condition and fixed strategies of the principal. After that, the investigations were focused on solving control problems (optimal contract design) under the conditions of participation and consistency. Subsequently, the emphasis was put on complex models describing multi-agent and dynamic organizational systems, contract renegotiation, and others (for an overview, see [8, 57]).

In the context of the insurance effect (i.e., risk redistribution), we should acknowledge the following conclusion drawn in contract theory. The difference between the efficient wage and the actual one is subject to that a risk-neutral principal (see Section 4.5) insures risk-averse agents against wage variations depending on the state of nature. Notably, wage stability is achieved due to the fact that in favorable[7] situations the reward appears smaller than the efficient wage. But on the other hand, in unfavorable situations the reward is higher than it would be without accounting for risk redistribution[8]. Let us provide an illustrative example.

Suppose an agent has two feasible actions, $A = \{y_1; y_2\}$, leading to two results $A_0 = \{z_1; z_2\}$; the corresponding probabilities are defined by the matrix $P = \begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$, where $1/2 < p \leq 1$. Hence, in the majority of cases (as far as $p > 1/2$) the result of agent's activity "coincides" with the corresponding action.

The costs incurred by choosing the first and the second actions constitute c_1 and c_2, respectively ($c_2 \geq c_1$). The expected income of the principal gained by the first (second) action makes H_1 (H_2, respectively). Next, the agent's reward for the first and second result of activity is σ_1 and σ_2, respectively. The principal's goal function Φ represents the difference between the income and the incentive. Finally, the agent's goal function f is the difference between the incentive and costs.

The principal's problem lies in assigning an incentive scheme to maximize the expected value of his or her goal function[9] $E\Phi$ (provided that the agent's action maximizes the expected value Ef of his or her own goal function).

Assume that the agent is *risk-neutral* (i.e., his or her *utility function* which reflects the attitude towards risk is linear). Consider the incentive scheme to-be-used by the principal for motivating the choice of the action y_1 by the agent. Under zero reservation utility, the problem of minimal incentive scheme implementing the action y_1 takes the form:

$$p\,\sigma_1 + (1-p)\,\sigma_2 \to \min_{\sigma_1 \geq 0,\, \sigma_2 \geq 0} \qquad (1)$$

[7] Activity of an enterprise (and the wages of its employees) depends on external macroparameters (seasonality, the periods of economic recession and upswing, world prices, etc.) and microparameters (health status of employees, and so on).

[8] Perhaps, exactly this important conclusion has an impact on further development of contract theory–most of models studied include only an external stochastic uncertainty. Indeed, in the deterministic case (or in the case of uncertainty with a risk-neutral agent), the insurance effects disappear and the actual wages coincides with the efficient one.

[9] Remind that the symbol "E" stands for the operator of mathematical expectation.

$$p\ \sigma_1 + (1-p)\ \sigma_2 - c_1 \geq p\ \sigma_2 + (1-p)\ \sigma_1 - c_2 \qquad (2)$$

$$p\ \sigma_1 + (1-p)\ \sigma_2 - c_1 \geq 0. \qquad (3)$$

The conditions (2)–(3) represent the incentive-compatibility constraint and the condition of agent's individual rationality, respectively.

The problem (1)–(3) is the one of linear programming.

In Figure 2.9, the set of rewards that satisfy the conditions (2)–(3) is shaded, while its subset with the minimum value of the expression (1) is marked by heavy line. The contour curve of the function (1)–see the dotted line in Figure 2.9–possesses the same slope as the segment[10] A_1B_1 (the arrow shows the direction of increase).

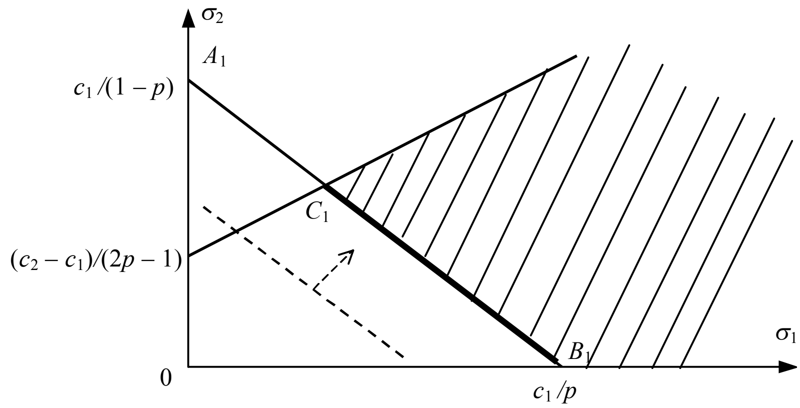

Figure 2.9. The principal implements the action y_1 under a risk-neutral agent.

Suppose the hypothesis of benevolence takes place. For definiteness, let us choose the point C_1 from the segment C_1B_1 as the corresponding solution; the point is given by

$$\sigma_1 = [p\,c_1 - (1-p)\,c_2]\ /\ (2p-1), \qquad (4)$$

$$\sigma_2 = [p\,c_2 - (1-p)\,c_1]\ /\ (2p-1). \qquad (5)$$

Evidently, the expected costs of the principal $E\sigma(y_1)$ to implement the action y_1 constitute c_1:

$$E\sigma(y_1) = c_1. \qquad (6)$$

Now, assume that the principal wants to implement the action y_2. By solving a problem similar to (1)–(3), we obtain the following (see the point C_2 in Figure 2.10):

[10] In the case of risk-neutral principal and agent, the existence of a solution set is typical in problems of contract theory. At the same time, a strictly concave utility function of the agent (which describes his or her risk-averse character) leads to uniqueness of the solution, see below.

$$\sigma_1 = [p\, c_1 - (1-p)\, c_2] \,/\, (2p - 1), \tag{7}$$

$$\sigma_2 = [p\, c_2 - (1-p)\, c_1] \,/\, (2p - 1), \tag{8}$$

$$E\sigma(y_2) = c_2. \tag{9}$$

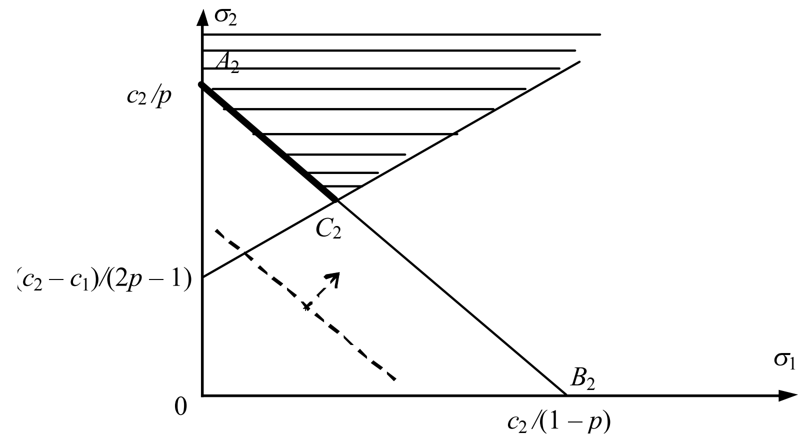

Figure 2.10. The principal implements the action y_2 under a risk-neutral agent/

At Step 2, the principal chooses what feasible action is more beneficial to implement, i.e., what action maximizes the difference between the principal's income and the expected costs to implement such action. Therefore, the expected value of the principal's goal function in the optimal contract equals

$$\Phi^* = max\ \{H_1 - c_1,\ H_2 - c_2\}.$$

To proceed, let us analyze the insurance effects in the model. Suppose that the agent appears risk-neutral; in other words, he or she estimates uncertain parameters according to a strictly increasing concave utility function $u(\cdot)$. Recall that the random variable (the result of agent's activity) determines his or her reward (the value of the incentive function). Thus, we believe that the agent's goal function is expressed as

$$f(\sigma(\cdot), z, y) = u(\sigma(z)) - c(y). \tag{10}$$

Introduce the substitutions[11] $v_1 = u(\sigma_1)$ and $v_2 = u(\sigma_2)$, where $u^{-1}(\cdot)$ is the inverse function to the agent's utility function (we assume it is nonnegative and vanishes in the origin).

Imagine the principal is interested to stimulate the agent's choice of the action y_1. Then the incentive problem takes the form

[11] Such change of variables makes it possible to linearize the system of constraints. It is used in the so-called two-step solution method for problems arising in contract theory [48].

$$p\, u^{-1}(v_1) + (1-p)\, u^{-1}(v_2) \to \min_{v_1 \geq 0,\, v_2 \geq 0} \quad (11)$$

$$p\, v_1 + (1-p)\, v_2 - c_1 \geq p\, v_2 + (1-p)\, v_1 - c_2, \quad (12)$$

$$p\, v_1 + (1-p)\, v_2 - c_1 \geq 0. \quad (13)$$

The conditions (12)–(13) represent the incentive-compatibility constraint and the condition of agent's individual rationality, respectively.

Clearly, the linear inequalities (12)–(13) coincide with (2)–(3) up to notation. Figure 2.11 below demonstrates the set of feasible values for the variables v_1 and v_2 (the shaded one). The contour curves of the function (11) (being convex due to concavity of the agent's utility function) are marked by the dotted line.

A strictly concave utility function of the agent leads to the strict convexity of the goal function (11). Consequently, the internal solution to the conditioned minimization problem (11)–(13) is unique. For instance, take the utility function $u(t) = \beta\, ln(1 + \gamma\, t)$, where β and γ are positive constants; hence, the solution is given by

$v_1 = c_1 + (c_1 - c_2)\,(1-p)\,/\,(2p-1)$, (14)

$v_2 = c_1 + (c_2 - c_1)\,p\,/\,(2p-1)$. (15)

Apparently, in this case the incentive scheme (14)–(15) makes the agent's expected utility (as the result of payments by the principal) equal to the agent's costs to choose the first action:

$$Ev = c_1. \quad (16)$$

Similarly, one may show that if the principal stimulates the agent to choose the second action, then the agent's expected utility (as the result of payments by the principal) coincides with the agent's costs to choose the second action.

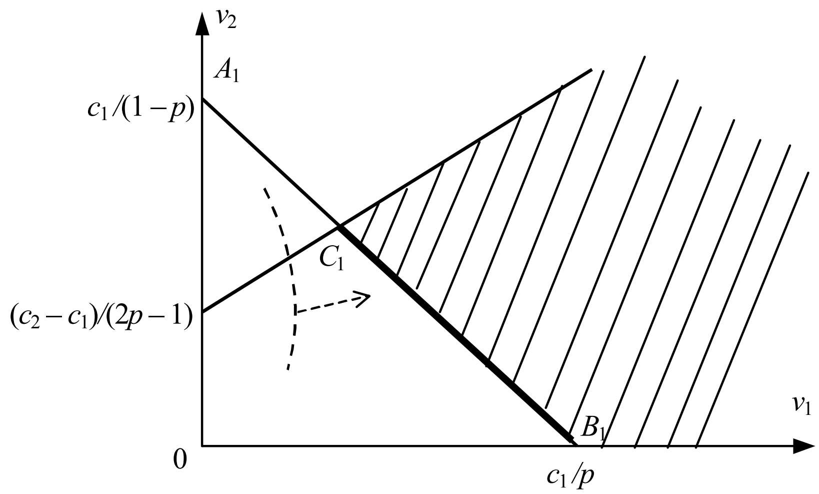

Figure 2.11. The principal implements the action y_1 under a risk-averse agent.

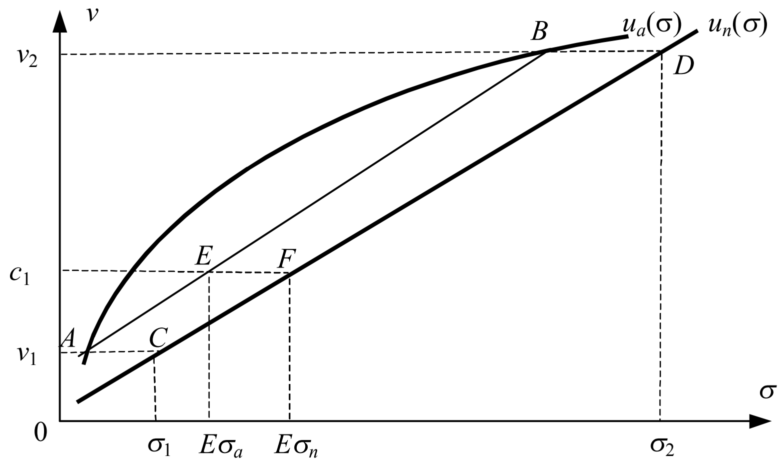

Figure 2.12. The insurance effect: the principal implements the action y_1

Formulas (14)–(15) imply that, in the case of a risk-averse agent, the principal "underpays" for implementation of the first result of activity ($v_1 \leq c_1$) and "overpays" for implementation of the second result of activity ($v_2 \geq c_1$). Moreover, in the limiting deterministic case[12] (as $p \to 1$) we have $v_1 \to c_1$.

The effect of insurance in the present model is illustrated by Figure 2.12 (implementation of the first action). Here the reader may find the linear utility function $u_n(\cdot)$ of the agent (defined up to an additive constant) and the strictly concave utility function $u_a(\cdot)$ of the agent. The segment *AB* lies to the top and/or to the left of the segment *CD*, and the expected utility in both cases makes c_1. Consequently, under a risk-averse agent the expected payments $E\sigma_a$ are smaller than the expected payments $E\sigma_n$ that correspond to a risk-neutral agent (compare the points *E* and *F* in Figure 2.12).

And so, we have analyzed an example of the insurance effect arising in the models of contract theory. Now, let us describe the incentive problems in multi-agent systems.

2.5. Collective Incentive Mechanisms

In the previous sections we have considered individual incentive systems. The present section and the three subsequent ones are dedicated to the description of *collective incentive systems*, i.e., incentives for a collective of agents.

An elementary extension of the basic single-agent model consists in a *multi-agent OS* with independent (noninteracting) agents. In this case, the incentive problem is decomposed into a set of corresponding single-agent problems.

Suppose that common constraints are imposed on the incentive mechanism for all agents or for a certain subset of agents. As the result, we derive the incentive problem in an *OS with*

[12] Note all models with uncertainty must meet the conformity principle: as the uncertainty vanishes (i.e., the limiting transition to a corresponding deterministic system takes place), all the results and estimates must tend to their deterministic counterparts. For instance, the expressions (14)–(15) for $p = 1$ determine optimal solutions in the deterministic case.

weakly related agents (discussed below). This problem represents a set of parametric single-agent problems, and one can search for optimal values of the parameters using standard techniques of constrained optimization.

If the agents are interrelated, *viz.*, the costs or/and rewards of an agent depend on his or her actions and the actions of the rest agents, one obtains a "full-fledged" multi-agent incentive model. It will be studied in the present section. Note this book provides no coverage of the situation when the same constraints apply to the sets of feasible states, plans or actions of the agents (for its detailed description, the reader is referred to [65]).

The sequences of solving the multi-agent and single-agent problems have much in common. At the beginning, it is necessary to construct a compensatory incentive scheme implementing a certain action (an arbitrary feasible action under given constraints). In fact, this is Stage 1–analyzing the incentive compatibility. Under the hypothesis of benevolence, in single-agent OS it suffices to verify that the maximum of the agent's goal function is attainable (by an implementable action). On the other hand, in multi-agent systems one should demonstrate that the choice of a corresponding action makes up an equilibrium strategy in the game of agents. Imagine there are several equilibria; then we have to verify the hypothesis of rational choice for an action in question. In the majority of cases, it takes only to accept the *unanimity axiom* (according to the latter, the agents do not choose equilibria being dominated in the sense of Pareto by other equilibria). Sometimes, the principal has to evaluate his or her guaranteed result on the set of equilibrium strategies of the agents, and so on. Further, it is necessary to equate the incentive and the costs and solve a standard optimization problem (find an implementable action to-be-rewarded by the principal). Actually, this is Stage 2–*incentive-compatible planning* (see Section 2.1). Let us study the above approach in a greater detail.

Incentives in OS with weakly related agents. Recall the results derived in Section 2.1 for the single-agent incentive problem. They can be directly generalized to the case of $n \geq 2$ agents if the following conditions hold true. The agent's goal functions depend only on their own actions (the case of the so-called *separable costs*), the incentive of each agent depends exclusively on his or her own actions, and some constraints are imposed on the total incentive of the agents. The formulated model is said to be an *OS with weakly related agents*. In fact, this is an intermediate case between individual and collective incentive schemes.

Let $N = \{1, 2, ..., n\}$ be a set of agents, $y_i \in A_i$ stand for an action of agent i, and $c_i(y_i)$ mean his or her costs. Moreover, denote by $\sigma_i(y_i)$ a reward given by a principal to agent i ($i \in N$); accordingly, $y = (y_1, y_2, ..., y_n)$ represents an action profile of the agents, $y \in A' = \prod_{i \in N} A_i$.

Assume that the principal gains the income $H(y)$ from agent's activity.

Suppose that the individual rewards of the agents are majorized by the quantities $\{C_i\}_{i \in N}$; in other words, $\forall\, y_i \in A_i$: $\sigma_i(y_i) \leq C_i$, $i \in N$. The *wage fund* (WF) being bounded by R (i.e., $\sum_{i \in N} C_i \leq R$), we obtain that the maximal set of implementable actions of agent i depends on the corresponding constraint R of the incentive mechanism. Within the framework of the assumptions of Section 2.1, this set makes up $P_i(C_i) = [0; y_i^+(C_i)]$, where

$$y_i^+(C_i) = max\ \{y \in A_i \mid c_i(y) \leq C_i\},\ i \in N.$$

Consequently, the optimal solution to the incentive problem in an OS with weakly related agents is defined as follows. One has to maximize the function

$$\Phi(R) = \max_{\{y_i \in P_i(C_i)\}_{i \in N}} H(y_1, ..., y_n)$$

by a proper choice of the individual constraints $\{C_i\}_{i \in N}$ that satisfy the *budget constraint* $\sum_{i \in N} C_i \leq R$. Apparently, this is a standard problem of constrained optimization.

We underline that for a fixed WF the agent's costs are not extracted from his or her income. At the same time, in the case of a variable WF, the optimal value R^* is a solution to the following optimization problem:

$$R^* = arg \max_{R \geq 0} [\Phi(R) - R].$$

Example 2.1. Choose the cost function of agent i as $c_i(y_i) = y_i^2 / 2r_i$, $i \in N$, and the principal's income function as $H(y) = \sum_{i \in N} \alpha_i y_i$, where $\{\alpha_i\}_{i \in N}$ are positive constants.

Under the imposed constraints $\{C_i\}_{i \in N}$, the maximal implementable action of each agent constitutes $y_i^+(C_i) = \sqrt{2r_i C_i}$, $i \in N$. The problem has been reduced to searching for an optimal set of the constraints $\{C_i^*\}_{i \in N}$ which meets the budget constraint and maximizes the principal's goal function:

$$\begin{cases} \sum_{i \in N} \alpha_i \sqrt{2r_i C_i} \to \max_{\{C_i \geq 0\}_{i \in N}} \\ \sum_{i \in N} C_i \leq R \end{cases}.$$

The unique solution to this problem has the form

$$C_i^* = \frac{r_i \alpha_i^2}{\sum_{j \in N} r_j \alpha_j^2} R, \; i \in I.$$

The optimal WF is $R^* = \sum_{i \in N} r_i \alpha_i^2 / 2$. •

Incentives in OS with strongly related agents. Denote by $y_{-i} = (y_1, y_2, ..., y_{i-1}, y_{i+1}, ..., y_n) \in A_{-i} = \prod_{j \neq i} A_j$ the opponents' action profile for agent i.

The preferences of the OS participants (the principal and the agents) are expressed by their goal functions. Notably, the principal's goal function $\Phi(\sigma, y)$ represents the difference between his or her income $H(y)$ and the total incentive $\upsilon(y)$ paid to the agents: $\upsilon(y) =$

$\sum_{i=1}^{n} \sigma_i(y)$, where $\sigma_i(y)$ is the reward of agent i, $\sigma(y) = (\sigma_1(y), \sigma_2(y), ..., \sigma_n(y))$. On the other part, the goal function of agent i, $f_i(\sigma_i, y)$, is determined by the difference between the reward obtained from the principal and the costs $c_i(y)$, namely,

$$\Phi(\sigma, y) = H(y) - \sum_{i=1}^{n} \sigma_i(y), \tag{1}$$

$$f_i(\sigma_i, y) = \sigma_i(y) - c_i(y),\ i \in N. \tag{2}$$

We acknowledge the following aspect. Both the individual incentive and the individual costs of agent i to choose the action y_i generally depend on the actions of all agents (*the case of weakly related agents with inseparable costs*).

Let us adopt the following sequence of moves in the OS. At the moment of their decision-making, the principal and agents know the goal functions and feasible sets of all OS participants. Enjoying the right of the first move, the principal chooses incentive functions and reports them to the agents. Next, under known incentive functions, the agents simultaneously and independently choose their actions to maximize appropriate goal functions.

We make a series of assumptions to-be-applied to different parameters of the OS:

1) for each agent, the set of feasible actions coincides with the set of nonnegative real values;
2) the cost functions of the agents are continuous and nonnegative; moreover, $\forall\ y_i \in A_i$: $c_i(y)$ does not decrease with respect to y_i and $\forall\ y_{-i} \in A_{-i}$: $c_i(0, y_{-i}) = 0$ ($i \in N$);
3) the principal's income function is continuous with respect to all arguments and attains the maximum for nonzero actions of the agents.

In essence, Assumption 2 implies that (regardless of the actions of the rest agents) any agent can minimize his or her costs by choosing an appropriate nonzero action. Other assumptions are similar to the single-agent case (see Section 2.1).

Both the costs and incentive of every agent generally depend on the actions of all agents. Hence, the agents are involved in a game, where the payoff of each agent depends on the actions of all the others. Suppose that $P(\sigma)$ is the set of equilibrium strategies of the agents under the incentive scheme $\sigma(\cdot)$ (in fact, this is the set of game solutions). For the time being, we do not specify the type of equilibrium, only presuming that the agents choose their strategies simultaneously and independently (thus, they do not interchange information and utility).

Similarly to the single-agent OS discussed in Section 2.1, guaranteed efficiency (or simply "*efficiency*") of an incentive scheme represents the minimal value (within the hypothesis of benevolence–the maximal value) of the principal's goal function over the corresponding set of game solutions:

$$K(\sigma) = \min_{y \in P(\sigma)} \Phi(\sigma, y). \quad (3)$$

The problem of *optimal incentive function/scheme design* lies in searching for a feasible incentive scheme σ^* yielding the maximal efficiency:

$$\sigma^* = arg \max_{\sigma \in M} K(\sigma). \quad (4)$$

The results of Section 2.1 imply the following. In the particular case of independent agents (i.e., the reward and costs of each agent are subject to his or her actions), the compensatory incentive scheme

$$\sigma_{iK}(y_i) = \begin{cases} c_i(y_i^*) + \delta_i, & y_i = y_i^* \\ 0, & y_i \neq y_i^* \end{cases}, \ i \in N, \quad (5)$$

appears optimal (to be correct, δ-optimal, where $\delta = \sum_{i \in N} \delta_i$). In the formulas above, $\{\delta_i\}_{i \in N}$ designate arbitrarily small strictly positive constants (bonuses). Moreover, the optimal action y^*, being implementable by the incentive scheme (5) as a *dominant strategy equilibrium*[13] (DSE), solves the following problem of optimal incentive-compatible planning:

$$y^* = arg \max_{y \in A'} \{H(y) - \sum_{i \in N} c_i(y_i)\}.$$

Suppose that the reward of each agent depends on the actions of all agents (this is exactly the case for collective incentives studied here) and the *costs are inseparable* (i.e., the costs of each agent generally depend on the actions of all agents, thus reflecting the interrelation of the agents). Then the sets of *Nash equilibria*[14] $E_N(\sigma) \subseteq A'$ and DSE $y_d \in A'$ take the form:

$$E_N(\sigma) = \{y^N \in A \mid \forall\, i \in N\ \forall\, y_i \in A_i, \quad (6)$$

$$\sigma_i(y^N) - c_i(y^N) \geq \sigma_i(y_i, y_{-i}^N) - c_i(y_i, y_{-i}^N)\}.$$

By definition, $y_{i_d} \in A_i$ is a dominant strategy of agent i iff

$$\forall\, y_i \in A_i,\ \forall\, y_{-i} \in A_{-i}:\ \sigma_i(y_{i_d}, y_{-i}) - c_i(y_{i_d}, y_{-i}) \geq \sigma_i(y_i, y_{-i}) - c_i(y_i, y_{-i}).$$

[13] Recall that a DSE is an action vector such that each agent benefits from choosing a corresponding component (irrespective of the actions chosen by the rest agents–see Appendix 1).

[14] Recall that a Nash equilibrium is an action vector such that each agent benefits from choosing a corresponding component provided that the rest agents choose equilibrium actions (see Appendix 1).

Imagine that a dominant strategy exists for each agent under a given incentive scheme. In this case, the incentive scheme is said to implement the corresponding action vector as a DSE.

Let us fix an arbitrary action vector $y^* \in A'$ of the agents and consider the following incentive scheme:

$$\sigma_i(y^*, y) = \begin{cases} c_i(y_i^*, y_{-i}) + \delta_i, & y_i = y_i^* \\ 0, & y_i \neq y_i^* \end{cases}, \ \delta_i \geq 0, \ i \in N. \quad (7)$$

In [50] it was shown that under the incentive scheme (7) used by the principal the vector y^* forms a DSE. Moreover, if $\delta_i > 0$, $i \in N$, then y^* makes up a unique DSE.

The incentive scheme (7) means that the principal adopts the following principle of decomposition. He or she suggests to agent i, "Choose the action y_i^*, and I compensate your costs regardless of the actions chosen by the rest agents. Yet, if you choose another action, the reward is zero." Using such strategy, the principal decomposes the game of agents.

Assume that the incentive of each agent depends implicitly only on his or her action. By fixing the opponents' action profile for each agent, let us pass from (7) to an individual incentive scheme. Notably, fix an arbitrary action vector $y^* \in A'$ of the agents and define the incentive scheme

$$\sigma_i(y^*, y_i) = \begin{cases} c_i(y_i^*, y_{-i}^*) + \delta_i, & y_i = y_i^* \\ 0, & y_i \neq y_i^* \end{cases}, \ \delta_i \geq 0, \ i \in N. \quad (8)$$

In this case, we have the following interpretation. The principal suggests to agent i, "Choose the action y_i^*, and I compensate your costs, as if the rest agents have chosen the corresponding actions y_{-i}^*. Yet, if you choose another action, the reward is zero." Adhering to such strategy, the principal also decomposes the game of agents, i.e., implements the vector y^* as a Nash equilibrium of the game.

Note that the incentive scheme (8) depends only on the action of agent i, while y_{-i}^* enters this function as a parameter. Moreover, in contrast to the incentive scheme (7), that of (8) provides each agent merely with indirect information about the action vector desired by the principal. For the incentive scheme (8) to implement the vector y^* as a DSE, additional assumptions should be introduced regarding the cost functions of the agents, see [65]. This is not the case for the incentive scheme (7).

It is not out of place to discuss here the role of the nonnegative constants $\{\delta_i\}_{i \in N}$ in the expressions (5), (7) and (8). If one needs implementing a certain action as a Nash equilibrium, these constants can be chosen zero. Imagine that the equilibrium must be unique (in particular, the agents are required not to choose zero actions; otherwise, in evaluation of the guaranteed result (3) the principal would be compelled to expect zero actions of the agents). In this case, the agents should be paid excess an arbitrarily small (strictly positive) quantity for choosing the action expected by the principal. Furthermore, the parameters $\{\delta_i\}_{i \in N}$ in formulas (5), (7) and (8) appear relevant in the sense of stability of the compensatory incentive scheme with respect to the model parameters. For instance, suppose that we know the cost function of agent i up to some constant $\Delta_i \le \delta_i / 2$. Consequently, the compensatory incentive scheme (7) still implements the action y^* (see [51]).

The vector of optimal implementable actions y^*, figuring in the expression (7) or (8) as a parameter, results from solving the following *problem of optimal incentive-compatible planning*:

$$y^* = arg \max_{t \in A'} \{H(t) - \upsilon(t)\}, \qquad (9)$$

where $v(t) = \sum_{i \in N} c_i(t)$, and the efficiency of the incentive scheme (7), (9) constitutes

$$K^* = H(y^*) - \sum_{i \in N} c_i(y^*) - \delta.$$

It was shown in [50] that the incentive scheme (7), (9) appears optimal, i.e., possesses the maximal efficiency among all incentive schemes in multi-agent OS.

Let us consider several examples of designing optimal collective incentive schemes in multi-agent OS.

Example 2.2. Solve the incentive problem in an OS with two agents, whose cost functions are $c_i(y) = \frac{(y_i + \alpha\, y_{3-i})^2}{2r_i}$, $i = 1, 2$; here α represents an interdependence parameter of the agents.

Assume that the principal's income function is defined by $H(y) = y_1 + y_2$, and the wage fund is bounded above by R. Under the incentive scheme (7) used by the principal, the incentive problem is to find optimal implementable actions:

$$\begin{cases} H(y) \to \max\limits_{y \ge 0} \\ c_1(y) + c_2(y) \le R \end{cases}.$$

Applying Lagrange's multiplier method yields the following solution:

$$y_1^* = \sqrt{\frac{2R}{r_1 + r_2}}\ \frac{\alpha\, r_2 - r_1}{\alpha^2 - 1},\ y_2^* = \sqrt{\frac{2R}{r_1 + r_2}}\ \frac{\alpha\, r_1 - r_2}{\alpha^2 - 1}.$$

Finally, substitute the equilibrium actions of the agents into the principal's goal function to obtain the optimal value of the WF:

$$R^* = arg \max_{R \ge 0} [\sqrt{2R(r_1 + r_2)} /(1 - \alpha) - R] = \frac{r_1 + r_2}{2(\alpha - 1)^2}.$$

Example 2.3. The second example is *the model of joint production*. Consider a multi-agent two-level OS composed of a principal and n agents.

Suppose that the goal function of agent i, $f_i\,(y,\ r_i)$, forms the difference between the income $h_i\,(y)$ gained by the joint activity and the costs $c_i\,(y,\ r_i)$, where r_i is the efficiency parameter (type) of the agent. In other words, $f_i\,(y,\,r_i) = h_i\,(y) - c_i\,(y,\,r_i),\ i \in N$.

Choose the following income and cost functions:

$$h_i\,(y) = \lambda_i\,\theta\,Y,\ i \in N,\ c_i\,(y,\,r_i) = \frac{y_i^2}{2(r_i \pm \beta_i \sum_{j \neq i} y_j)},\ i \in N,$$

where $Y = \sum_{i \in N} y_i$, $\sum_{i \in N} \lambda_i = 1$. We believe that $\sum_{j \neq i} y_j < \frac{r_i}{\beta_i}$ for the case of minus in the denominator.

A possible interpretation lies in a firm which manufactures the same product sold at a price θ on the market. The total income $\theta\,Y$ is distributed among the agents according to fixed shares $\{\lambda_i\}_{i \in N}$. For each agent, the costs increase with respect to his or her actions, while the agent's type r_i determines the efficiency of activity.

Interaction of the agents is modeled by the relationship between the costs (the efficiency of activity) of each agent and actions of the rest agents.

The sign "+" in the denominator corresponds to the efficient interaction of the agents (decreasing costs). Indeed, the larger actions are chosen by the rest agents, the smaller costs has the agent in question (the higher is the efficiency of his or her activity). In practice, this means the reduction of fixed incremental charges, sharing of experience or technologies, etc.

On the other hand, the sign "–" in the denominator describes the inefficient interaction of the agents (increasing costs). The larger actions are chosen by the rest agents, the higher costs has the agent in question (accordingly, his or her efficiency drops). In practice, such situation corresponds to the shortage of the basic assets, constraints imposed on secondary indicators (e.g., environmental pollution) and so on.

The coefficients $\{\beta_i \geq 0\}_{i \in N}$ reflect the interdependence level of the agents.

Suppose that all OS participants know the market price θ. Involve the first-order necessary optimality conditions for the agents' goal functions to obtain

$$y_i = \lambda_i\,\theta\,(r_i \pm \beta_i \sum_{j \neq i} y_j),\ i \in N.$$

By summing up the above expressions, we have the following relationship between the total actions Y^+ and the parameter θ:

$$Y^+(\theta) = \frac{\sum_{i \in N} \frac{\lambda_i\,\theta\,r_i}{1 \pm \lambda_i \theta \beta_i}}{1 \mp \sum_{i \in N} \frac{\lambda_i\,\theta\,\beta_i}{1 \pm \lambda_i \theta \beta_i}}.$$

In fact, incentives mean modifying the parameters $\{\lambda_i\}_{i \in N}$, that represent internal (corporate, transfer) prices.

Example 2.4. The third example is *lump-sum labor payments*. Consider an OS with two agents, whose cost functions are $c_i(y_i) = y_i^2 / 2r_i$ (r_i stands for the type of agent i, $y_i \in A_i = \Re_1^+$, $i = 1, 2$). The goal function of agent i is given by the difference between the reward $\sigma_i(y_1, y_2)$ paid by a principal and the costs, namely,

$$f_i(y) = \sigma_i(y) - c_i(y_i),\ i = 1, 2.$$

Let the principal adopt the incentive scheme

$$\sigma_i(y_1, y_2) = \begin{cases} C_i, & y_1 + y_2 \geq x \\ 0, & y_1 + y_2 < x \end{cases},\ i = 1, 2. \quad (10)$$

Thus, the principal provides a fixed incentive to each agent if their total action is not smaller than a planned value $x > 0$. Denote by $y_i^+ = \sqrt{2r_iC_i}$, $i = 1, 2$, $Y = \{(y_1, y_2) \mid y_i \leq y_i^+,$ $i = 1, 2,\ y_1 + y_2 \leq x\}$, the set of individually rational actions of the agents. We study four possible combinations of the variables (see Figures 2.13–2.16).

In case 1 (Figure 2.13), the set of Nash equilibria forms the segment $E_N(\sigma) = [N_1; N_2]$. Fix an arbitrary equilibrium $y^* = (y_1^*, y_2^*) \in E_N(\sigma)$. Hence, the set of Nash equilibria possesses the cardinality of continuum; this causes some disadvantages in the sense of efficiency of incentives, see below.

All points of the segment $[N_1; N_2]$ are Pareto-efficient from the agents' point of view. Accordingly, it seems rational to pay excess the agents for choosing specific actions from the segment (a certain strictly positive small incentive).

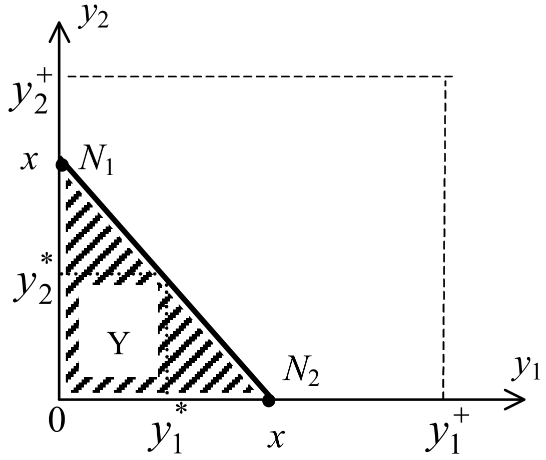

Figure 2.13.

We design an individual incentive scheme according to the results derived above (see formulas (8)-(9)):

$$\tilde{\sigma}_1^*(y_1) = \sigma_1(y_1,\ y_2^*) = \begin{cases} C_1,\ y_1 \geq y_1^* \\ 0,\quad y_1 < y_1^* \end{cases}, \tag{11}$$

$$\tilde{\sigma}_2^*(y_2) = \sigma_2(\ y_1^*, y_2) = \begin{cases} C_2,\ y_2 \geq y_2^* \\ 0,\quad y_2 < y_2^* \end{cases}.$$

Under such incentive scheme, the point $y^* = (\ y_1^*,\ \ y_2^*)$ is a unique Nash equilibrium. Notably, by passing from the incentive scheme (10) for each agent (this scheme depends on the actions of all agents) to the incentive scheme (11) (which is completely defined by the action of the agent in question), the principal decomposes the game of agents and implements the unique action. Evidently, the efficiency of the incentive does not decrease; quite the reverse, it can be higher than for the initial incentive scheme.

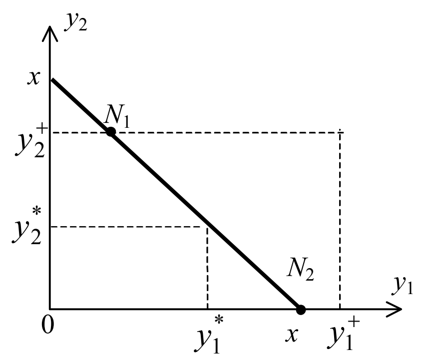

Figure 2.14.

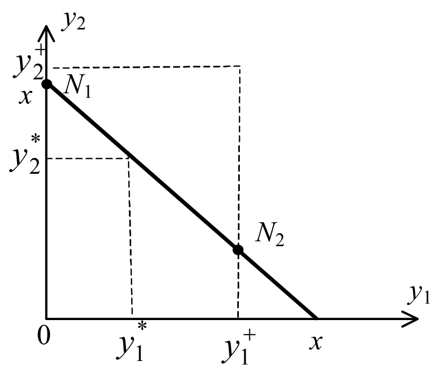

Figure 2.15.

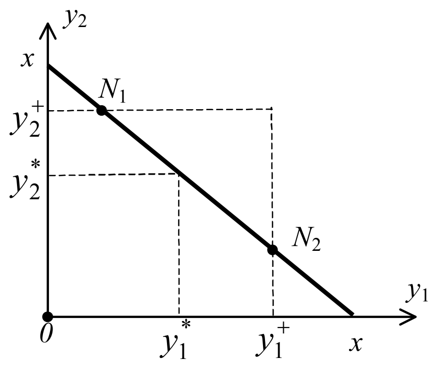

Figure 2.16.

In cases 2 and 3, Nash equilibria are the segments [N_1; N_2] illustrated by Figs. 2.14 and 2.15, respectively.

Finally, in case 4 (see Figure 2.16), the set of Nash equilibria consists of the point (0; 0) and the segment [N_1; N_2], i.e.,

$$E_N(\sigma) = (0; 0) \cup [N_1; N_2].$$

Moreover, the points of the interval (N_1; N_2) are (different) undominated equilibria in the sense of Pareto.

Now, within the framework of the current example, let the cost functions of the agents be inseparable:

$$c_i(y) = \frac{(y_i + \alpha\, y_{3-i})^2}{2r_i}.$$

We define the set of individually rational actions of the agents: $Y = \{(y_1, y_2) \mid c_i(y) \le C_i,\ i = 1, 2\}$. To avoid consideration of all feasible combinations of the parameters $\{r_1, r_2, C_1, C_2, x\}$, we take the case demonstrated in Figure 2.17.

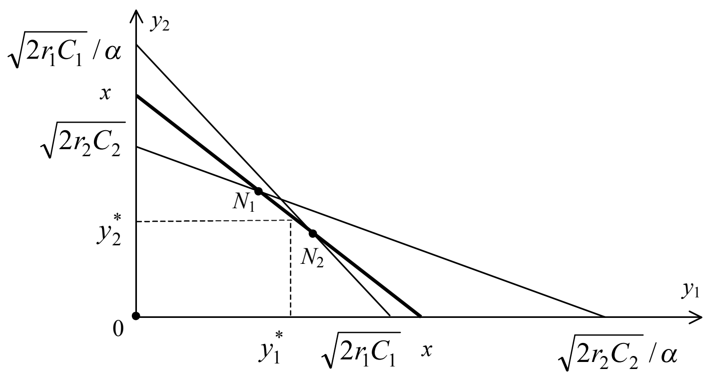

Figure 2.17. The set of Nash equilibria [N_1; N_2] under inseparable costs.

Consequently, the set of Nash equilibria includes the segment $[N_1; N_2]$. The incentive scheme

$$\tilde{\sigma}_1^*(y) = \begin{cases} c_1(y_1^*, y_2), & y_1 = y_1^* \\ 0, & y_1 \neq y_1^* \end{cases} \tag{12}$$

$$\tilde{\sigma}_2^*(y) = \begin{cases} c_2(y_1, y_2^*), & y_2 = y_2^* \\ 0, & y_2 \neq y_2^* \end{cases}$$

implements the action $y^* \in [N_1; N_2]$ as a dominant strategy equilibrium.

We have finished discussion of the incentive mechanisms for individual results of agents' activity. To proceed, let us describe some collective incentive mechanisms.

2.6. Collective Incentive Mechanisms

The majority of known incentive models consider two types of OS. The first type is when a control subject (a principal) observes the result of activity for all controlled subjects (agents), being uniquely defined by the strategy (action) chosen by an agent. The second type includes OS with uncertainties, where the observed result of agents' activity depends not only on his or her actions, but also on uncertain and/or random factors (e.g., see the model of contract theory in Section 2.4).

The present section provides the statement and solution to the collective incentive problem in a multi-agent deterministic OS, where a principal possesses only some aggregated information about the results of agents' activity.

Recall the model studied in the preceding section. In an n-agent OS, let the *result of agents' activity* $z \in A_0 = Q(A')$ be a certain function of their actions: $z = Q(y)$ ($Q(\cdot)$ is referred to as the *aggregation function*). The preferences of the OS participants, i.e., the principal and agents, are expressed by their goal functions. In particular, the principal's goal function makes up the difference between his or her income $H(z)$ and the total incentive $\upsilon(z)$ paid to the agents: $\upsilon(z) = \sum_{i \in N} \sigma_i(z)$, where $\sigma_i(z)$ stands for the incentive of agent i, $\sigma(z) = (\sigma_1(z), \sigma_2(z), \ldots, \sigma_n(z))$, i.e.,

$$\Phi(\sigma(\cdot), z) = H(z) - \sum_{i \in N} \sigma_i(z). \tag{1}$$

The goal function of agent i represents the difference between the reward given by the principal and the costs $c_i(y)$:

$$f_i(\sigma_i(\cdot), y) = \sigma_i(z) - c_i(y),\ i \in N. \tag{2}$$

We adopt the following sequence of moves in the OS. At the moment of decision-making, the principal and the agents know the goal functions and feasible sets of each other, as well as the aggregation function. The principal's strategy is choosing incentive functions, while the agents choose their actions. Enjoying the right of the first move, the principal chooses incentive schemes and report them to the agents. Under known incentive functions, the agents subsequently choose their actions by maximizing the corresponding goal functions.

Imagine that the principal observes individual actions of the agents (equivalently, the principal can uniquely recover the actions using the observed result of activity). In this case, the principal may employ an incentive scheme being directly dependent on the agents' actions: $\forall\ i \in N$: $\tilde{\sigma}_i(y) = \sigma_i(Q(y))$ - how such incentive problems are treated is discussed in the previous section. Therefore, we analyze a situation when the principal observes merely the result of activity in the OS (which predetermines the principal's income); he or she is unaware of the individual actions of the agents and appears unable to restore this information. In other words, *aggregation of information* takes place–the principal possesses incomplete information on the action vector $y \in A'$ of the agents. He or she only knows a certain aggregated variable $z \in A_0$ (a parameter characterizing the results of joint actions of the agents).

In the sequel, we believe that the OS parameters meet the assumptions introduced in Section 2.5. Moreover, assume that the aggregation function is a one-valued continuous function.

By analogy to the aforesaid, the efficiency of incentive is treated as the minimal value (or the maximal value–under the hypothesis of benevolence) of the principal's goal function on the corresponding solution set of the game:

$$K(\sigma(\cdot)) = \min_{y \in P(\sigma(\cdot))} \Phi(\sigma(\cdot), Q(y)). \tag{3}$$

The problem of optimal incentive function design lies in searching for a feasible incentive scheme σ^* ensuring the maximal efficiency:

$$\sigma^* = \arg \max_{\sigma(\cdot)} K(\sigma(\cdot)). \tag{4}$$

In the incentive problems investigated in Section 2.5, decomposition of the agent game was based on the principal's ability to motivate the agents for choosing a specific (observable!) action. Actions of the agents being unobservable, direct application of the decomposition approach seem impossible. Thus, solution of the incentive problems (where the agents' rewards depend on the aggregated result of activity in the OS) should follow another technique.

This technique is rather transparent. Find a set of actions yielding a given result of activity. Then separate a subset with the minimal total costs of the agents (accordingly, with the minimal costs of the principal to stimulate the agents under optimal compensatory incentive functions, see Sections 2.1 and 2.5). Next, construct an incentive scheme implementing this subset of actions. Finally, choose the result of activity with the most beneficial implementation for the principal.

Now, let us give a formal description to the solution of the incentive problem in an OS with aggregation of information about agents' activity.

Define the set of action vectors of the agents, leading to a given result z of activity:

$$Y(z) = \{y \in A' \mid Q(y) = z\} \subseteq A',\ z \in A_0.$$

It has been demonstrated above that, in the case of observable actions of the agents, the minimal costs of the principal to implement the action vector $y \in A'$ equal the total costs of the agents $\sum_{i \in N} c_i(y)$. Similarly, we evaluate the minimal total costs of the agents to demonstrate the result of activity $z \in A_0$: $\tilde{\vartheta}(z) = \min_{y \in Y(z)} \sum_{i \in N} c_i(y)$, and the corresponding action set $Y^*(z) = Arg \min_{y \in Y(z)} \sum_{i \in N} c_i(y)$, which attains the minimum.

Fix an arbitrary result of activity $x \in A_0$ and an arbitrary vector $y^*(x) \in Y^*(x) \subseteq Y(x)$. Let us make a technical assumption as follows: $\forall\, x \in A_0$, $\forall\, y' \in Y(x)$, $\forall\, i \in N$, $\forall\, y_i \in Proj_i\, Y(x)$: the function $c_j\,(y_i,\, y'_{-i})$ does not decrease with respect to y_i, $j \in N$. It was rigorously shown in [50] that:

1) under the incentive scheme

$$\sigma^*_{ix}(z) = \begin{cases} c_i(y^*(x)) + \delta_i, & z = x \\ 0, & z \neq x \end{cases},\ i \in N, \tag{5}$$

the action vector of the agents $y^*(x)$ is implementable as a unique equilibrium with the minimal costs of the principal to stimulate the agents (these costs constitute $\tilde{\vartheta}(x) + \delta$, $\delta = \sum_{i \in N} \delta_i$);

2) the incentive scheme (5) is δ-optimal.

Hence, Step 1 to solve the incentive problem (4) consists in finding the minimal incentive scheme (5), which leads to the principal's costs $\tilde{\vartheta}(x)$ to stimulate the agents and implements the action vector of the agents, leading to the given result of activity $x \in A_0$. Therefore, at Step 2 one evaluates the most beneficial (for the principal) result of activity $x^* \in A_0$ by solving the problem of optimal incentive-compatible planning:

$$x^* = arg \max_{x \in A_0} [H(x) - \tilde{\vartheta}(x)]. \tag{6}$$

And so, the expressions (5)–(6) provide the solution to the problem of optimal incentive scheme design in the case of agents' joint activity.

Let us analyze how the principal's ignorance (infeasibility of observations) of the agents' actions affects the efficiency of incentives. As usual, it is assumed that the principal's income function depends on the result of activity in the OS. Consider two possible cases.

1. the actions of the agents are observable, and the principal is able to motivate the agents based on their actions and on the result of collective activity;
2. the actions of the agents are unobservable, and the incentives could depend on the observed result of collective activity (exclusively).

Let us compare the efficiency of incentives in these cases.

Under observable actions of the agents, the principal's costs $\vartheta_1(y)$ to implement the action vector $y \in A'$ of the agents constitute $\vartheta_1(y) = \sum_{i \in N} c_i(y)$, and the efficiency of incentives is $K_1 = \max_{y \in A'} \{H(Q(y)) - \vartheta_1(y)\}$ (see Section 2.5, as well).

Actions of the agents being unobserved, the minimal costs of the principal $\vartheta_2(z)$ to implement the result of activity $z \in A_0$ are defined by (see (5)-(6)): $\vartheta_2(z) = \min_{y \in Y(z)} \sum_{i \in N} c_i(y)$; accordingly, the efficiency of incentives makes $K_2 = \max_{z \in A_0} \{H(z) - \vartheta_2(z)\}$.

In [50] it was demonstrated that $K_1 = K_2$. The described phenomenon could be referred to as the *perfect aggregation theorem* for incentive models. In addition to the estimates of comparative efficiency, it has an extremely important methodological sense. It turns out that, under collective incentive scheme, the principal ensures the same level of efficiency as in the case of individual incentive scheme!

In other words, aggregation of information by no means decreases the operational efficiency of an organizational system. This sounds somewhat paradoxically, since existing uncertainties and aggregation generally reduce the efficiency of management decisions. The model considered includes *perfect aggregation*. In practice, the interpretation is that the principal does not care what actions are selected by the agents; they must lead to the desired result of activity provided the minimum total costs. Informational load on the principal is decreased (provided the same efficiency of incentives).

Therefore, the performed analysis yields the following conclusions. If the principal's income depends only on the aggregated indicators of agents' activity, using them is reasonable to motivate the agents. Even if individual actions of the agents are observed by the principal, an incentive scheme based on individual actions of the agents does not increase the efficiency of control (but definitely raises informational load on the principal).

Recall that in Section 2.1 we have formulated the principle of costs compensation. For models with data aggregation, the principle is extended in the following way. The minimal costs of the principal to implement a given result of activity in the OS are defined as the minimal total costs of the agents compensated by the principal (provided that the former choose an action vector leading to this result of activity). Let us consider an illustrative example.

Example 2.5. Set $z = \sum_{i \in N} y_i$, $H(z) = z$, $c_i(y_i) = y_i^2 / 2r_i$, $i \in N$ (see also the examples in Section 2.5). We evaluate

$$Y(z) = \{y \in A' \mid \sum_{i \in N} y_i = z\}.$$

Solution to the problem

$$\sum_{i \in N} c_i(y_i) \to \min_{y \in A'} \text{ under the constraint } \sum_{i \in N} y_i = x$$

takes the form $y_i^*(x) = \frac{r_i}{W} x$, where $W = \sum_{i \in N} r_i$, $i \in N$. The minimal costs to implement the result of activity $x \in A_0$ are equal to $\vartheta(x) = x^2 / 2W$.

By maximizing the principal's goal function (by estimating $\max_{x \geq 0} [H(x) - \vartheta(x)]$), we obtain the optimal plan $x^* = W$ and the optimal incentive scheme

$$\sigma_i^*(W, z) = \begin{cases} r_i \frac{x^2}{2W^2}, & z = x \\ 0, & z \neq x \end{cases}, \; i \in N.$$

The efficiency of incentives (the value of the principal's goal function) is $K = W/2$.

In Sections 2.5-2.6 we have studied collective incentive schemes with specific (for each agent) relationships between the rewards and actions or results of the agents. In practice, it happens that the principal has to apply the same relationship between the rewards and actions (or results of joint activity) for all agents. To proceed, let us focus on such models.

2.7. UNIFIED INCENTIVE MECHANISMS

In *personalized* (individual and collective) incentive schemes studied above, for any agent the principal chooses a specific relationship between the reward and his or her actions (Section 2.1), the actions of the rest agents (Section 2.5) or the results of their joint activity (Section 2.6). In addition to personalized incentive schemes, there exist *unified* incentive schemes with an identical (for all agents) relationship between the reward and certain parameters. The necessity of using unified incentives follows from institutional constraints or emerges as the result of principal's aspiration for "democratic-type" management, suggesting equal opportunities for the agents, etc.

Since unified control is a special ("simplified") case of personalized control, the efficiency of the former is not greater than that of the latter. Hence, the following questions arise. What efficiency losses appear as the result of egalitarianism? When the losses actually vanish?

Consider two models of collective unified incentives, *viz*., a unified proportional (linear) incentive scheme and a unified collective incentive scheme for the results of joint activity; note this technique could be used for the analysis of any incentive scheme. In the first model, unification leads to no efficiency losses (just the opposite, unified incentive schemes turn out optimal in the class of all proportional ones). In the second model, the efficiency is considerably lower.

Unified proportional incentive schemes. Let us introduce the following assumption regarding the cost functions of the agents:

$$c_i(y_i, r_i) = r_i\, \varphi(y_i / r_i),\ i \in N, \tag{1}$$

where $\varphi(\cdot)$ is a smooth strictly increasing convex function such that $\varphi(0) = 0$ (e.g., for the Cobb-Douglas function we have $\varphi(t) = t^{\alpha} / \alpha,\ \alpha \geq 1$), and $r_i > 0$ is the efficieny parameter (*type*) of agent *i*.

Suppose that the principal uses proportional (L-type) individual incentive schemes: $\sigma_i(y_i) = \gamma_i\, y_i$. Then the agents' goal functions take the form $f_i(y_i) = \gamma_i\, y_i - c_i(y_i)$. Find the action chosen by agent *i* under a certain fixed incentive scheme applied by the principal:

$$y_i^*(\gamma_i) = r_i\, \varphi'^{-1}(\gamma_i),\ i \in N, \tag{2}$$

with $\varphi'^{-1}(\cdot)$ being the reverse function to the derivative of $\varphi(\cdot)$.

The minimal total costs of the principal to stimulate the agents constitute

$$\vartheta_L(\gamma) = \sum_{i=1}^{n} \gamma_i\, r_i\, \varphi'^{-1}(\gamma_i), \tag{3}$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$.

The total costs of the agents make up

$$c(\gamma) = \sum_{i=1}^{n} r_i\, \varphi(\varphi'^{-1}(\gamma_i)). \tag{4}$$

Note that the above general model of proportionnal incentives covers different statements of specific problems. We study some statements below, interpreting actions of the agents as the amounts of products manufactured.

Problem 1. Suppose that the principal is interested in agents' performing an assigned plan R of the total output under the minimum costs of the agents. We underline the necessity of distinguishing the total costs of the agents and the total costs of the principal to motivate the agents. Then he or she chooses wage rates $\{\gamma_i\}_{i \in N}$ by solving the following problem:

$$\begin{cases} c(\gamma) \to \min\limits_{\gamma} \\ \sum\limits_{i=1}^{n} y_i^*(\gamma_i) = R \end{cases}. \tag{5}$$

This problem has the following solution:

$$\gamma_i^* = \varphi'(R / W);\ y_i^* = r_i\,(R / W);\ i \in N,$$

$$c^* = W\,\varphi\,(R / W);\ \vartheta_L^* = R\,\varphi'(R / W), \tag{6}$$

where $W = \sum\limits_{i=1}^{n} r_i$.

Since optimal wage rates turn out identical for all agents, exactly the unified linear incentive scheme is optimal.

Problem 2. The problem of total output maximization under the constraints on the total costs of the agents

$$\begin{cases} \sum\limits_{i=1}^{n} y_i^*(\gamma_i) \to \max\limits_{\gamma} \\ c(\gamma) \le R \end{cases}, \tag{7}$$

is "dual" to Problem 1. The solution to (7) is given by:

$$\gamma_i^* = \varphi'(\varphi^{-1}(R / W));\ y_i^* = r_i\,\varphi^{-1}(R / W);\ i \in N,$$

$$c^* = R;\ \vartheta_L^* = \varphi^{-1}(R / W)\ W\,\varphi'(\varphi^{-1}(R / W)). \tag{8}$$

In other words (and naturally enough), the optimal solution again consists in unified proportional incentive schemes.

Now, in Problems 1 and 2 substitute the total costs of the agents by the total costs of the principal to motivate them. This generates another pair of dual problems.

Problem 3. Suppose that the principal is interested in agents' implementation of a planned total output R under the minimum total costs to motivate them. The corresponding wage rates are defined by solving the following problem:

$$\begin{cases} \vartheta_L(\gamma) \to \min\limits_{\gamma} \\ \sum\limits_{i=1}^{n} y_i^*(\gamma_i) = R \end{cases}. \tag{9}$$

Then the unified scheme (6) again provides the solution (exactly as in Problem 1). This is rather a curious result, since the total costs of the agents reflect the interests of the controlled subjects, while the total costs to motivate them correspond to the interests of the principal. Of course, the reason lies in the assumptions made earlier.

Problem 4 is to maximize the total output under existing constraints on the total costs to motivate the agents:

$$\begin{cases} \sum_{i=1}^{n} y_i^*(\gamma_i) \to \max_{\gamma} \\ \vartheta_L(\gamma) \le R \end{cases}. \tag{10}$$

Lagrange's multiplier method yields the following optimality condition (λ is the Lagrange multiplier):

$$\lambda\, \varphi'^{-1}(\gamma_i)\, \varphi''(\gamma_i) + \gamma_i = 1,\ i \in N.$$

Hence, all wage rates must be identical and satisfy the equation

$$\gamma\, \varphi'^{-1}(\gamma) = R / W. \tag{11}$$

Therefore, we have derived the following result. In organizational systems with weakly related agents (whose cost functions are given by (1)), unified incentive schemes turn out optimal on the set of proportional incentive schemes.

Note that the optimality of unified proportional incentive schemes (*UL-type incentive schemes*) could also be demonstrated on the set of proportional incentive schemes in OS with weakly related agents and the cost functions (1). Thus, it seems interesting to study their relative efficiency on the set of all possible (not only proportional) incentive schemes. It suffices to compare the minimal costs to motivate the agents, e.g., in Problem 2, with the costs to motivate the agents in the case of the optimal compensatory incentive schemes $\vartheta_K(y^*) = \sum_{i=1}^{n} r_i\, \varphi(y_i / r_i)$ (see Sections 2.1 and 2.5).

By solving the choice problem for the vector $y^* \in A'$ which maximizes $\vartheta_K(y^*)$ under the constraint $\sum_{i=1}^{n} y_i^* = R$, one obtains that $\vartheta_K^* = W\, \varphi(R / W)$. By substituting $\vartheta_{UL}^* = R\, \varphi'(R / W)$ from the expression (6), one evaluates the ratio of the minimal costs to motivate the agents:

$$\vartheta_{UL}^* / \vartheta_K^* = R / W\, \varphi'(R / W) / \varphi(R / W). \tag{12}$$

From the convexity of $\varphi(\cdot)$ it follows that $\vartheta_{UL}^* / \vartheta_K^* \ge 1$. Moreover, one would easily show that for $R / W > 0$ and strictly convex cost functions the ratio (12) exceeds the unity. The total costs to motivate the agents in unified proportional schemes are higher than in the case of “absolutely optimal” compensatory incentive schemes. Accordingly, the former are nonoptimal in the class of all feasible (e.g., nonnegative and monotonic) incentive schemes. The result obtained for multi-agent organizational systems agrees with the conclusion made in Section 2.3 (in single-agent systems the efficiency of proportional incentives does not exceed that of compensatory incentives).

Unified incentive schemes for joint activity results. In Section 2.5 we have analyzed personalized incentive schemes of agents for the results of their joint activity. Now, let us apply a unified incentive scheme to this model.

Consider the class of unified incentive schemes for joint activity results (see Section 2.5), i.e., the incentive schemes where a principal uses an identical relationship between individual rewards and the result of activity $z \in A_0$ for all the agents. We introduce the following function:

$$c(y) = \max_{i \in N} \{c_i(y)\}. \qquad (13)$$

At Step 1, evaluate the minimal costs $\vartheta_U(z)$ of the principal to implement the result of activity $z \in A_0$ under a unified incentive scheme:

$$\vartheta_U(z) = \min_{y \in Y(z)} c(y).$$

The set of action vectors minimizing the costs to implement the result of activity $z \in A_0$ takes the form

$$Y^*(z) = Arg \min_{y \in Y(z)} c(y).$$

By analogy to Section 2.5, one may show that the unified incentive scheme

$$\sigma_{ix}(z) = \begin{cases} c(y^*(x)) + \delta / n, & z = x \\ 0, & z \neq x \end{cases}, \ i \in N, \qquad (14)$$

where $y^*(x)$ is an arbitrary element from the set $Y^*(x)$, guarantees the *incentive compatibility* and implements the result of activity $x \in A_0$ under the minimal principal’s costs (in the class of unified incentive schemes).

At Step 2 of designing the optimal unified incentive scheme, one evaluates the most beneficial (according to the principal’s viewpoint) result of activity x_U^* in the OS. For this, solve the *problem of optimal incentive-compatible planning*:

$$x_U^* = arg \max_{z \in A_0} [H(z) - n\, \vartheta_U(z)]. \qquad (15)$$

Formulas (14)–(15) describe the solution to the problem of optimal unified incentive scheme design in the case of joint activity of the agents. Obviously, the efficiency of the unified incentives (14)–(15) does not exceed the efficiency of the personalized incentives (5)–(6).

Example 2.6. Recall the first example in Section 2.5 and assume that the principal has to use a unified incentive scheme. Set $c(y) = y_j^2/2r_j$, where $j = arg \min_{i \in N} \{r_i\}$. Then the minimal costs to motivate the agents constitute $\vartheta_U(z) = z^2/2\, n\, r_j$. The optimal plan $x_U^* = n\, r_j$ yields the efficiency $n\, r_j / 2$. Generally, it is smaller than the efficiency $\sum_{i \in N} r_i / 2$ ensured by the personalized incentive scheme (both efficiencies coincide in the case of identical agents).

2.8. TEAM INCENTIVE MECHANISMS

This section is devoted to the description of collective incentive models, notably, *team payments*. They are remarkable for that an agent (a member of a team) obtains a reward defined by *activity participation factor* (APF); thus, the reward depends on the action of the agent in comparison with the actions of other agents. Generally, the bonus fund is determined according to an aggregated activity result of the whole team (in a particular case, it is fixed).

It is possible to apply different APF design procedures:

- making APF proportional to a *wage category* (qualification level) of an employee;
- making APF proportional to *activity contribution factor* (ACF) of an employee (his or her individual BSC).

Forming APF in a proportion to wage categories means the following. Suppose that a wage category characterizes the activity of each employee (agent). Moreover, the greater is the wage category, the higher is the qualification level of an agent. Hence, a wage category describes the efficiency of each agent and can be involved to assess his or her activity.

In the case of ACF, one accounts for the actual contribution of each agent to the overall result of the whole team (depending on individual labor productivity and quality of work).

Thus, in a work team the managers possess specific goals and form conditions of functioning to achieve them. Accordingly, the agents have their own goals and strive for attaining them by a proper choice of actions.

We believe that, based on the results of its activity, a work team gains a given bonus fund R to-be-distributed among the agents under a chosen incentive scheme.

Suppose that agent i is assigned a rate r_i reflecting his or her qualification level (the efficiency of activity) and the individual costs of agent i, $c_i = c_i(y_i, r_i)$, strictly decrease with respect to the qualification level r_i, $i \in N$. A work team composed of agents with an identical qualification level is said to be *uniform* (and *non-uniform* if the levels differ). In the sequel, the efficiency of an incentive scheme is defined by the total action of the agents: $\Phi(y) = \sum_{i \in N} y_i$.

The procedures based on APF. First, let us consider the case of APF. A fund R is distributed among the agents according to the activity participation coefficients $\{\delta_i\}_{i \in N}$, $\sum_{j \in N} \delta_j = 1$. Thus, the bonus of agent i makes up $\sigma_i = \delta_i R$.

The goal functions of the agents take the form

$$f_i(y_i) = \sigma_i - c_i(y_i, r_i),\ i \in N. \qquad (1)$$

Due to its simplicity, a widespread procedure of APF design is based merely on accounting for the qualification level of agent i, i.e., $\delta_i = \frac{r_i}{\sum_{j \in N} r_j}$. Substitute this formula in (1) to obtain the following result. APF involving solely the qualification levels of the agents (and not their actual actions) exert no impact on the agents. In particular, such procedures do not motivate the agents to choose, e.g., larger actions. Therefore, we proceed to ACF.

The procedures based on ACF. For an agent, a natural and elementary technique to define ACF is making it proportional to the action of the agent:

$$\delta_i = \frac{y_i}{\sum_{j \in N} y_j},\ i \in N. \qquad (2)$$

Suppose that the cost functions of the agents are linear: $c_i(y_i, r_i) = y_i / r_i$. Then the expressions (1)-(2) imply that the goal function of agent i depends on the actions of all agents:

$$f_i(y) = R\ \delta_i = \frac{y_i}{\sum_{j \in N} y_j} - y_i / r_i,\ i \in N. \qquad (3)$$

Hence, the modeled situation represents a game among n players with the payoff functions (3).

Uniform teams. We begin with uniform teams (with identical agents). Nash equilibrium actions of the agents take the form:

$$y_i^* = \frac{Rr(n-1)}{n^2},\ i \in N, \qquad (4)$$

leading to the following value of the *efficiency*:

$$K_1(R, r, n) = \frac{Rr(n-1)}{n}. \qquad (5)$$

Formula (4) demonstrates that a larger bonus fund stimulates the choice of greater actions by the agents. According to (5), the efficiency grows linearly as one increases the bonus fund or qualification levels of the agents. In other words, there is no optimal bonus fund which maximizes the efficiency K_1 / R of its utilization. At the same time, the actions of the agents being bounded above, there exists an optimal value of the bonus fund. If one knows the upper bound, the optimal value is expressed from (4). It is easy to prove that partitioning a uniform team in smaller subteams (with appropriate splitting up of the bonus fund) would not increase the efficiency of its utilization. Moreover, for a fixed wage fund any reduction of a uniform team causes efficiency losses and the choice of larger actions by the agents.

Consider the following issue. Preserving the same bonus fund R, is it possible to improve the total efficiency of a uniform team by a proper design of agents' ACF?

For this, let us study the following ACF design procedure:

$$\delta_i = \frac{y_i^\alpha}{\sum_{j \in N} y_j^\alpha},\ i \in N,\ 1 \le \alpha \le \frac{n}{n-1}. \tag{6}$$

Note it turns out more sensitive to agents' differentiation than the procedure (2). Nash equilibrium actions of the agents are

$$y_i^* = \alpha \frac{Rr(n-1)}{n^2},\ i \in N, \tag{7}$$

and appear greater than the actions given by (4).

Hence, under the constraint $1 \le \alpha \le \frac{n}{n-1}$, one claims that applying the ACF design procedure (6) improves the efficiency as against the procedure (2) by $1 / (n - 1)$ per cent. For instance, in a team of 11 employees, the possible gain constitutes 10%.

Non-uniform teams. Formulas (2)-(3) show that, in the corresponding non-uniform team, the Nash equilibrium is ensured by the following agents' actions and efficiency[15]:

$$y_i^* = \frac{\sum_{j \in N} 1/r_j - (n-1)/r_i}{(\sum_{j \in N} 1/r_j)^2} R(n-1),\ i \in N, \tag{8}$$

$$K_2(R,\ \vec{r},\ n) = \sum_{j \in N} y_j^* = \frac{R(n-1)}{\sum_{j \in N} 1/r_j}. \tag{9}$$

[15] For identical agents, (8) and (4), as well as (9) and (5) coincide.

Assume that a team includes agents of two types, i.e., m skillful agents with the efficiency r^+ and $(n – m)$ "ordinary" agents with the efficiency $r^–$, where $r^+ > r^–$.

Then one derives $\sum_{i \in N} 1/r_i = m / r^+ + (n – m) / r^–$.

Using the expression (8), find the equilibrium actions of the skillful agents:

$$y^+ = \frac{R(n-1)}{m/r^+ + (n-m)/r^-}\left[1 - \frac{1}{r^+}\frac{(n-1)}{m/r^+ + (n-m)/r^-}\right], \quad (10)$$

and of the "ordinary" agents:

$$y^- = \frac{R(n-1)}{m/r^+ + (n-m)/r^-}\left[1 - \frac{1}{r^-}\frac{(n-1)}{m/r^+ + (n-m)/r^-}\right]. \quad (11)$$

Next, evaluate the efficiency by formula (9):

$$K_2(R, m, n) = \frac{R(n-1)}{m/r^+ + (n-m)/r^-}. \quad (12)$$

Evidently, (8) and (10)-(11) demonstrate that the agents with a higher level of qualification compel the agents with a lower qualification level to choose smaller actions. Consequently, the values of their goal functions are accordingly decreased.

In addition, formula (11) indicates of the following. If the number of skillful agents in a team is such that $m \geq \frac{1/r^-}{1/r^- - 1/r^+}$, then the "ordinary" agents benefit nothing by increasing their actions. Yet, for $m = 1$, the "ordinary" agents always benefit from increasing their actions. At the same time, simple reasoning shows that skillful agents improve the efficiency of the whole team (despite the choice of small actions by the "ordinary" agents).

Let us analyze the feasibility of further increase in the efficiency in a team under a fixed bonus fund R. Partition a non-uniform team into two uniform subteams. Suppose that the first subteam includes m skillful agents, and $(n – m)$ "ordinary" agents form the second one. We also accordingly divide the bonus fund R of the team: $R = R^+ + R^–$. In the Nash equilibrium, the efficiency of the first (second) subteam equals $\frac{R^+ r^+ (m-1)}{m}$ (respectively, $\frac{R^- r^- (n-m-1)}{n-m}$).

Hence, the overall efficiency of the team of n agents is

$$K_3(R, m, n) = \frac{R^+ r^+ (m-1)}{m} + \frac{R^- r^- (n-m-1)}{n-m}. \quad (13)$$

We have noted earlier that partitioning a uniform team into several subteams by no means improves the total efficiency. Generally, this is not true for a non-uniform team. For instance, compare the expressions (12) and (13) provided that the team includes a half of skillful agents whose efficiency exceeds by two times the efficiency of "ordinary" agents; then separating the first group in a subteam would increase the total efficiency only if the initial team is composed of six agents maximum. Otherwise, reduction of the total efficiency is feasible as the result of partitioning the non-uniform team into two uniform subteams (even in the case of optimal distribution of the bonus fund among the subteams).

Individual and collective incentives. To conclude the current section, we compare the efficiency of individual and collective incentives in a series of practically relevant situations.

Let the cost functions of the agents be linear: $c_i (y_i, r_i) = y_i / r_i$, $i \in N$. Assume there exists a common constraint y^{max} for the maximum actions of agents: $A_i = [0; y^{max}]$, $i \in N$.

Renumber the agents in the descending order of the efficiencies of their activity:

$$r_1 \geq r_2 \geq \ldots \geq r_n. \tag{14}$$

Suppose that y^{max} is such that the action y_1^* defined by (8) under $i = 1$ appears feasible. In this case, the actions of the rest agents are also feasible under the incentive scheme (2) based on ACF. The efficiency of team incentive $K_2(R, \vec{r}, n)$ is given by formula (9).

Evaluate the efficiency of individual incentive when the principal may motivate the agents independently for individual results of activity (provided that the total incentive does not exceed R). We adopt the principle of costs compensation (see Section 2.1) and the results obtained for the incentive problem in a system with weakly related agents (see Section 2.5).

If the principal uses compensatory incentive schemes, it is optimal to compensate the costs of the first k agents in the sequence (14) (alternatively, the costs of the first (k + 1) agents–depending on the parameters):

$$k = min\ \{j \in N \mid y^{max} \sum_{i=1}^{j} 1/r_i \leq R, y^{max} \sum_{i=1}^{j+1} 1/r_i > R\}. \tag{15}$$

Formula (15) means that the principal should first employ the agents whose efficiency is maximal. In other words, a nonzero reward is provided to the first k or (k + 1) agents, while the rest receive nothing (employing them seems unreasonable). Thus, the efficiency of individual incentive is

$$K_4(R, \vec{r}, n) = k\, y^{max} + r^{k+1}\ (R - y^{max} \sum_{i=1}^{k} 1/r_i\ ). \tag{16}$$

The expressions (9) and (16) serve for analyzing the efficiencies gained by collective and individual incentives.

As a rule, individual incentives are more efficient (see Section 2.7). For instance, in the case of uniform teams we have the estimate:

$K_4(R, r, n) / K_1(R, r, n) \approx n / (n - 1) \geq 1$.

Team payments are close to the so-called *rank incentive schemes*, where collective rewards involve the procedures of competition, norms, etc. A detailed discussion of this class of collective incentive schemes could be found in [50] and Section 2.10. In what follows, we analyze incentive mechanisms in matrix structures.

2.9. INCENTIVE MECHANISMS IN MATRIX STRUCTURES

In many real organizational systems, the same agent is simultaneously subordinated to several principals located at the same or at different hierarchical levels. The first case is said to be *distributed control*, while the second one is known as *interlevel interaction*.

Interlevel interaction. The analysis of interlevel interaction models [50] indicates that double subordination of an agent to principals located at different levels of an hierarchy appears inefficient. An indirect confirmation of this fact consists in a famous management principle "a vassal of my vassal is not my vassal." Therefore, naturally each agent should be subordinated only to his or her immediate superior, a principal located exclusively at the next (higher) level of an hierarchy (see also the models of hierarchies optimization in [44]).

The following question arises accordingly. Why in real organizational systems one often observes the effects of interlevel interaction? A possible descriptive explanation (without consideration of normative interaction structure for the participants and institutional constraints) is given below. Generally, efficiency losses are assumed to appear only due to the factors of aggregation, decomposition of control problems and insufficient awareness of a principal about certain agents. In particular, imagine there are informational constraints at an intermediate level (e.g., the amount of information to-be-processed by a principal in a certain system exceeds his or her capabilities). Then some control functions (possibly, in the aggregated form) are transferred to a higher level. Simply speaking, incompetency of an intermediate principal (in the objective sense) usually represents the primary cause of the interlevel interaction observed in practice. Therefore, on the one hand, one should a priori admit the feasibility of interlevel interaction in solving the design problems for institutional, functional, informational and other structures in an OS (still, striving to avoid it as much as possible). On the other hand, the presence of interlevel interaction in a real OS testifies to its nonoptimal functioning; for a manager, this is an indication of the necessity to review the structure (or even the composition) of the system.

At the same time, double subordination of agents to some same-level principals may be unavoidable. An example is matrix control structures being remarkable for distributed control.

Distributed control. A specific feature of *matrix control structures* (MCS) is that a single employee appears simultaneously subordinated to several superiors (at the same hierarchical level) performing different functions (e.g., a coordinating function, a supporting function, a controlling function, and others). Note that MCS are natural for project-oriented organizations. For instance, a "horizontal" structure of projects is superimposed on an hierarchical organizational structure (see Figure 2.18).

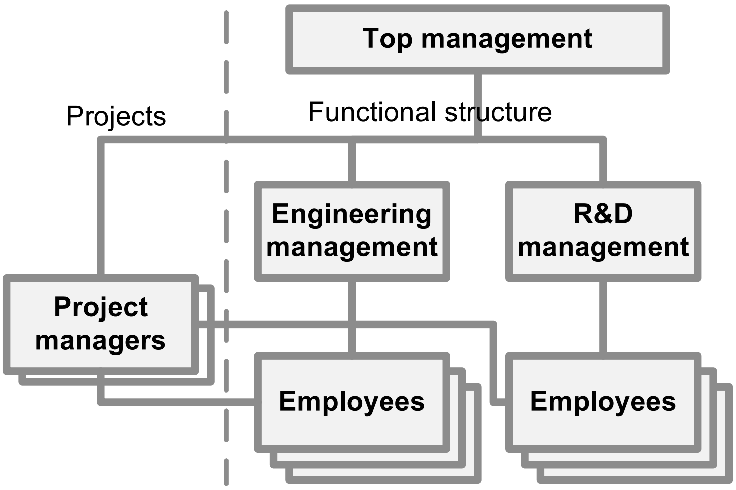

Figure 2.18. A matrix control structure of an organization.

In MCS, principals controlling an agent are involved in a "game," with a complicated equilibrium. In particular, we can separate two stable modes of interaction among the principals–the cooperation mode and the competition mode.

In the *cooperation mode*, the principals act jointly and such strategies ensure the required results of agent's activity under minimal resources.

In the *competition mode* (note it takes place when the goals of the principals differ appreciably), the resources are spent inefficiently.

Let us formulate an elementary model of a matrix control structure; a comprehensive overview of modern research in the field of these control problems could be found in [12, 50].

Suppose that an OS consists of a single agent and k principals. A strategy of the agent lies in choosing an action $y \in A$, which incurs the costs $c(y)$. As the result of the agent's activity, each principal gains a certain income described by the function $H_i(y)$. Moreover, principal i pays to the agent the reward $\sigma_i(y)$, $i \in K = \{1, 2, \ldots, k\}$ (K stands for the set of principals). Thus, the goal function of principal i takes the form

$$\Phi_i(\sigma_i(\cdot), y) = H_i(y) - \sigma_i(y),\ i \in K. \tag{1}$$

The agent's goal function is defined by

$$f(\{\sigma_i(\cdot)\}, y) = \sum_{i \in K} \sigma_i(y) - c(y). \tag{2}$$

The sequence of moves is the following. The principals simultaneously and independently choose incentive functions and report them to the agent; the latter then chooses an action.

For the game of principals, let us confine the analysis to the set of Pareto-efficient Nash equilibria. In this case, the principals' strategies are

$$\sigma_i(x, y) = \begin{cases} \lambda_i, & y = x \\ 0, & y \neq x \end{cases}, \ i \in K. \qquad (3)$$

This means the principals agree about motivating the agent's choice of a specific action $x \in A$ (referred to as a *plan*), as well as agree about implementation of joint incentives. Such mode of interaction among the principals is said to be the cooperation mode.

The conditions of optimality (in the Pareto sense) imply the following. The total incentive received by the agent from the principals (in the case of plan fulfillment) equals the costs:

$$\sum_{i \in K} \lambda_i = c(x). \qquad (4)$$

In fact, this is the principle of costs compensation generalized to the systems with distributed control.

For each principal, the beneficial cooperation condition may be stated as follows. In the cooperation mode, each principal obtains the utility not smaller than as if he or she motivated the agent independently (by compensating the agent's costs to choose the most beneficial action for this principal). The utility of principal i from "independent" interaction with the agent is defined by (see Section 2.1):

$$W_i = \max_{y \in A} \ [H_i(y) - c(y)], \ i \in K. \qquad (5)$$

Denote $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and let

$$S = \{x \in A \mid \exists \, \lambda \in \Re_+^k : H_i(x) - \lambda_i \geq W_i, \ i \in K, \ \sum_{i \in K} \lambda_i = c(x)\} \qquad (6)$$

be the set of agent's actions such that their implementation makes cooperation of the principals beneficial.

A set of the pairs $x \in S$ and corresponding vectors λ is called a *domain of compromise*:

$$\Lambda = \{x \in A, \ \lambda \in \Re_+^k \mid H_i(x) - \lambda_i \geq W_i, \ i \in K, \ \sum_{i \in K} \lambda_i = c(x)\}. \qquad (7)$$

By definition, the cooperation mode takes place if the domain of compromise is non-empty: $\Lambda \neq \varnothing$. In the cooperation mode the agents obtain zero utility.

Set

$$W_0 = \max_{y \in A} \ [\sum_{i \in K} H_i(y) \ - c(y)]. \qquad (8)$$

Then the domain of compromise is non-empty iff [50]

$$W_0 \geq \sum_{i \in K} W_i \,. \tag{9}$$

Thus, the implementability criterion for the cooperation mode is given by formula (9). This means that, acting jointly, the principals may gain a greater total efficiency as against their single-handed behavior. The difference $W_0 - \sum_{i \in K} W_i$ can be interpreted as a measure of interests' coordination among the principals and as a rate of OS emergence.

The condition (9) being not met ($\Lambda = \varnothing$), the competition mode takes place for the principals; it is characterized by the so-called *auction solution*. Sort (renumber) the principals in the ascending order of the quantities $\{W_i\}$: $W_1 \geq W_2 \geq \ldots \geq W_k$. The winner is the first principal suggesting the agent a utility being by an arbitrarily small quantity greater than W_2 (provided that the agent's costs are compensated, as well).

Let us discuss the obtained results. A drawback of MCS is that under insufficient separation of authorities a conflict may take place between the principals–e.g., project managers and functional managers; both project managers and their functional collegues (i.e., principals located at an intermediate level of an hierarchy) strive for "winning over" the agents being simultaneously controlled by them. Evidently, the whole OS bears certain losses of efficiency, since gaining the support of the agents may require considerable costs.

Cooperation of middle-level principals (joint assignment of plans and application of an agreed incentive scheme of the form (3) to motivate the agents) enables avoiding such conflicts and efficiency losses. Passing from the competition mode to the cooperation mode makes it necessary to coordinate the interests of the principals. This can be done by superiors at higher hierarchical levels via certain incentive techniques. We consider a possible model[16] below.

Earlier, we have analyzed the cases when in an MCS middle-level principals of an hierarchy (e.g., project managers) benefit from cooperation. The principals form a coalition and jointly assign a plan for an agent. Hence, all principals can be treated as a single agent maximizing the goal function

$$\Phi_K(\cdot) = \sum_{i \in K} H_i(y) - c(y) \,. \tag{10}$$

However, is such situation good or bad for a *top manager* (TM) (see Figure 2.18) representing the interests of the whole organization? Answering the posed question requires defining the interests of TM, as well as the methods of influencing on operation of the organizational system.

According to the TM viewpoint, the controlled subject is the set of all middle-level principals and the agent. The principals are described by the income functions $H_i(y)$, $i \in K$, while the agent is described by the cost function $c(y)$.

Suppose that the interests of a principal depend only on the result of system functioning, i.e., on the values of income and costs implemented by the agent's actions. Then the TM goal function is given by $F(\cdot) = F(H_1(\cdot),\ldots, H_k(\cdot), c(\cdot))$.

[16] This model has been developed by M.V. Goubko, Cand. Sci. (Tech.).

Moreover, it seems rational to assume the following. The TM aims for increasing (as much as possible) the income gained by each project (represented by the agents) and for reducing the costs to implement these projects. Thus, the TM goal function increases with respect to the variables $H_1, H_2, \ldots, H_k$ and decreases with respect to the agent's costs c.

An elementary goal function of TM is a linear convolution of all subgoals (with certain nonnegative weights α_i), yielding the criterion

$$F(y) = \sum_{i \in K} \alpha_i H_i(y) - \alpha_0 c(y) . \tag{11}$$

Compare this formula with the expression (10) defining the goal function in the case of a colition of the principals. Apparently, all the coefficients $\{\alpha_i\}$ being different, the system includes interests' miscoordination between the TM and middle-level principals (project managers). By maximizing their goal function, the latter implement a "wrong" action of the agent (undesired by the TM). Hence, the TM should exert an impact on middle-level principals for reducing the gap between the implemented action y and the required one (which maximizes the efficiency criterion (11)).

A possible technique of influencing the system functioning by the TM consists in internal "taxation." Notably, certain deduction rates $\{\beta_i\}$ for benefit of the TM are established for the incomes gained by the principals of the middle level $\{H_i(\cdot)\}$. Furthermore (or alternatively), certain deduction rates γ_i can be introduced for the profits $\{H_i(\cdot) - \sigma_i(\cdot)\}$. We will show that complete agreement between the interests of the TM and middle-level principals takes place in the case of a flat rate $\gamma \in [0; \gamma_{max}]$ being applied to the profits of all principals.

Under a flat rate for profit tax and a differential rate for income tax, the goal function of the TM and of the middle-level principals' coalition are respectively rewritten as[17]

$$F(y) = \gamma[\sum_{i \in K} \alpha_i \beta_i H_i(y) - \alpha_0 c(y)] \tag{12}$$

and

$$\Phi(y) = (1-\gamma)[\sum_{i \in K} (1-\beta_i) H_i(y) - c(y)] . \tag{13}$$

To coordinate the interests of the TM and middle-level principals, it suffices that their goal functions attain the maximum at the same point. Formulas (12)-(13) imply that this condition holds true if $\alpha_i \beta_i / \alpha_0 = 1 - \beta_i$ (i.e., under the income tax rate $\beta_i = \dfrac{1}{1 + \alpha_i / \alpha_0}$).

The TM is interested in increasing the share in the profits; thus, $\gamma = \gamma_{max}$. In such taxation system, complete interests' coordination between the TM and project managers (middle-level

[17] Note that in the models (11) and (12) the interests of TM differ.

principals) takes place. For instance, the income tax rate is 50% provided that $\alpha_i = 1$ ($i \in K$) and $\alpha_0 = 0$.

Therefore, in multi-level organizational systems, ensuring efficient operation requires that each higher level of the hierarchy performs interests' coordination with lower-level agents (particularly, by the choice of an appropriate incentive scheme). In other words, normal operation of an MCS requires that the top manager uses control actions such that middle-level principals can elaborate joint policy and assign coordinated plans to the agents.

2.10. RANK INCENTIVE MECHANISMS

In many incentive schemes, rewards of agents depend on the absolute values of their actions (see Section 2.1) and/or on the result of activity (see Sections 2.5, 2.7 and 2.8). At the same time, *rank incentive schemes* (RIS) are widely adopted in practice; an agent's reward is defined either by his or her activity indicator (the action or the result) belonging to a certain given range of values (*normative RIS*), or by the number the agent keeps in the ordered sequence of activity indicators of the agents (*competition RIS*).

A major advantage of rank incentive schemes lies in that the principal not necessarily needs to know actual actions chosen by the agents. Instead, he or she suffices to know feasible ranges the actions belong to or information on the ordered actions.

Normative RIS (NRIS) are remarkable for the existing procedures of assigning certain ranks to the agents depending on their activity indicators (e.g., selected actions, etc.). Let us introduce the following assumptions to-be-valid in the current section.

First, suppose that the sets of feasible actions of the agents are identical and coincide with the set A of nonnegative real values. Second (similarly to Sections 2.1 and 2.5), assume that the cost functions of the agents are monotonic and vanish in the origin (choosing zero action leads to zero costs).

Denote by $N = \{1, 2, \ldots, n\}$ the set of agents, by $\Im = \{1, 2, ..., m\}$ the set of feasible ranks, where m is the dimension of an NRIS. Next, $\{q_j\}$, $j = \overline{1, m}$, represent a certain set of m nonnegative values of the rewards for corresponding ranks. Finally, δ_i: $A_i \to \Im$, $i = \overline{1, n}$, are classification procedures. Then an NRIS is the tuple $\{m, \Im, \{\delta_i\}, \{q_j\}\}$.

For any incentive scheme, there is a corresponding NRIS having not smaller efficiency. Indeed, for any incentive scheme and any agent, one may find an individual procedure of classifying his or her actions, so as under NRIS the agent chooses the same action as under the initial incentive scheme. However, in practice it appears unreasonable (or even impossible) to use an individual classification procedure for each agent. Therefore, consider the case of an identical classification procedures adopted for all agents (known as a *unified* NRIS (UNRIS). The problems of incentive scheme identification are also discussed in Section 2.7.

Unified normative rank incentive schemes. A UNRIS being applied, the agents choosing the same actions obtain the same rewards. Consider a vector $Y = (Y_1, Y_2, ..., Y_m)$ such that $0 \leq Y_1 \leq Y_2 \leq ... \leq Y_m < +\infty$; it defines a certain partition of the set A. A unified NRIS is determined by the tuple $\{m, \{Y_j\}, \{q_j\}\}$, provided that the reward of agent i (denoted by σ_i) constitutes $\sigma_i(y_i) = \sum_{j=0}^{m} q_j I(y_i \in [Y_j, Y_{j+1}))$, where $I(\cdot)$ means the indicator function, $Y_0 = 0$, $q_0 = 0$.

A unified NRIS is said to be *progressive* if the rewards increase with respect to the actions: $q_0 \le q_1 \le q_2 \le ... \le q_m$. The curve of a progressive UNRIS is shown in Figure 2.19.

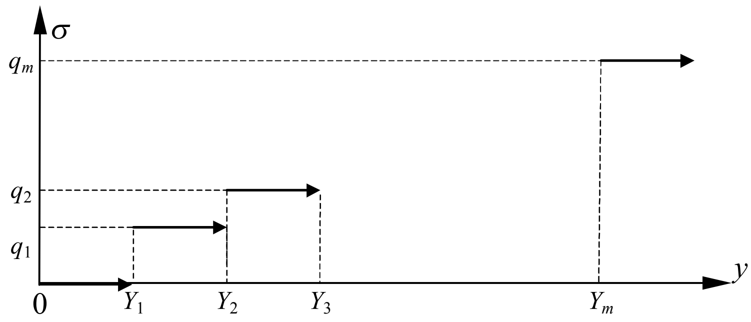

Figure 2.19. An example of a progressive UNRIS.

A UNRIS is a piecewise constant function. Hence, it follows from the monotonic property of the cost functions that the agents choose actions with minimal costs on the corresponding segments. In other words, one may believe that under a fixed incentive scheme the set of feasible actions is $Y = \{Y_1, Y_2, ..., Y_m\}$, and $q_0 = 0$ if $c_i(0) = 0$. The action y_i^* chosen by agent i depends on the pair of vectors (Y, q), i.e., $y_i^*(Y, q) = Y_{k_i}$, where

$$k_i = arg \max_{k=0,m} \{q_k - c_i(Y_k)\}, i \in N. \tag{1}$$

Set $y^*(Y, q) = (y_1^*(Y, q), y_2^*(Y, q), ..., y_n^*(Y, q))$. The problem of optimal UNRIS design lies in choosing a UNRIS dimension m and vectors q, Y satisfying the given constraints and maximizing the principal's goal function:

$$\Phi(y^*(Y, q)) \to \max_{Y,q} . \tag{2}$$

Fix a certain action vector $y^* \in A' = A^n$ desired by the principal as the result of UNRIS implementation.

Recall that within a UNRIS the agents choose actions from the set Y. Therefore, the minimal dimension of the incentive scheme must be equal to the number of pairwise different components of the action vector to-be-implemented. Consequently, using a UNRIS of a greater dimension than n seems unreasonable. We confine ourselves with incentive schemes whose dimension coincides with the number of agents: $m = n$.

Given a fixed action vector $y^* \in A'$, set $Y_i = y_i^*$, $i \in N$, and denote $c_{ij} = c_i(Y_j)$, $i, j \in N$. The definition of an implementable action (see (1)) implies the following. A necessary and sufficient condition for a UNRIS to implement the action vector $y^* \in A'$ (motivating the agents to choose the corresponding actions) consists in the following system of inequalities:

$$q_i - c_{ii} \geq q_j - c_{ij},\ i \in N, j = \overline{0, n}. \tag{3}$$

Let

$$\vartheta(y^*) = \sum_{i=1}^{n} q_i(y^*) \tag{4}$$

be the total costs to implement the action y^* in a UNRIS; here $q(y^*)$ satisfies the system (3).

The problem of optimal (minimal) UNRIS design is to minimize (4) under the constraints (3).

Suppose that the agents can be sorted in the ascending order of their costs and marginal costs:

$$\forall\, y \in A:\ c_1'(y) \geq c_2'(y) \geq \ldots \geq c_n'(y).$$

To proceed, fix a certain vector $y^* \in A'$ such that

$$y_1^* \leq y_2^* \leq \ldots \leq y_n^*. \tag{5}$$

Notably, the higher are the agent's costs, the smaller actions he or she actually chooses.

The described assumptions hold true for many common cost functions of the agents widely used in mathematical economics.

For instance, these are $c_i(y_i) = k_i\, c(y_i)$, $c_i(y_i) = k_i\, c(y_i / k_i)$, $c(0) = 0$, where $c(\cdot)$ indicates a monotonic differentiable function and the coefficients are sorted: $k_1 \geq k_2 \geq \ldots \geq k_n$ (they reflect the efficiency of agents' activity). Special cases are linear cost functions, the Cobb-Douglas cost functions and others.

In [50] it was shown that:

1) unified normative rank incentive schemes implement only the actions meeting the condition (5);
2) optimal UNRIS is progressive;
3) optimal rewards can be evaluated using the recurrent formula $q_1 = c_{11}$, $q_i = c_{ii} + \max_{j<i} \{q_j - c_{ij}\}$, $i = \overline{2, n}$;
4) in a UNRIS implementing the vector $y^* \in A'$, the individual rewards satisfy

$$q_i = \sum_{j=1}^{i} (c_j(y_j^*) - c_j(y_{j-1}^*)). \quad (6)$$

The expression (6) serves for analyzing the properties of a UNRIS. It is possible to find optimal rewards, to construct optimal classification procedures, to compare the efficiency of a UNRIS with the efficiency of compensatory incentive schemes, etc.

Competition rank incentive schemes. Let us briefly consider some properties of *competition rank incentive schemes* (CRIS), where a principal defines the number of classes and the number of available places within each class, as well as the rewards of the agents in a certain class. In other words, in a CRIS an individual reward of an agent does not directly depend on the absolute value of his or her action. Instead, the reward is determined by the place the agent has according to the ordered actions or results of activity of all agents. It was proved in [50] that:

1) the inequality (5) is a necessary and sufficient condition of implementability of the action vector $y^* \in A'$ in the class of CRIS;
2) the above vector is implementable by the following incentive scheme ensuring the minimal costs of the principal:

$$q_i(y^*) = \sum_{j=2}^{i} \{c_{j-1}(y_j^*) - c_{j-1}(y_{j-1}^*)\}, \; i = \overline{1,n}. \quad (7)$$

Formula (7) is useful to study the properties of a CRIS (to find optimal rewards, to construct optimal classification procedures, to compare the efficiency of a CRIS with the efficiency of other incentive schemes, etc.).

2.11. MECHANISMS OF ECONOMIC MOTIVATION

Incentive mechanisms motivate controlled subjects (agents) to perform specific actions for the benefit of a control subject (a principal). In the previous sections of this chapter, we have discussed the mechanisms, where an incentive lies in a direct reward of an agent by the principal. In contrast, this section focuses on certain *mechanisms of economic motivation*–a principal controls agents by establishing different *norms* (e.g., tax rates, profit rates, and so on) of agents' activity. The corresponding examples are internal taxation rates defining income or profit allocation between units or departments and the whole organization (a corporate principal or a holding company) and external taxation rates defining the payments of enterprises to regional or municipal budgets.

Consider the following model. An organizational system (a corporation, a firm) includes a principal and n agents. Suppose the costs $c_i(y_i)$ of agent i to be known and dependent on his or her action $y_i \in \Re^1_+$ (e.g., on the amount of products manufactured by the agent); $i \in N = \{1, 2, \ldots, n\}$, where N is the set of agents. Moreover, assume that the cost function is continuous, increasing and convex, vanishing in the origin. The goal function of agent i represents the difference between his or her income $H_i(y_i)$ and the costs $c_i(y_i)$:

$f_i(y_i) = H_i(y_i) - c_i(y_i),\ i \in N.$

Let the cost functions of the agents take the form

$c_i(y_i) = r_i\ \varphi(y_i / r_i),\ i \in N,$

where $\varphi(\cdot)$ is an increasing smooth convex function, $\varphi(0) = 0$.

Denote by $\xi(\cdot) = \varphi'^{-1}(\cdot)$ the inverse function to the derivative $\varphi(\cdot)$.

In the sequel, we study five mechanisms of economic motivation of the agents, *viz*.,

1) deduction mechanism (for income tax);
2) centralized mechanism;
3) profitability rate mechanism;
4) profit tax mechanism;
5) profit share mechanism.

Deduction mechanism. Suppose that an internal (transfer) price λ is given for a unit product manufactured by agents. Imagine that a principal uses a certain *deduction rate*[18] $\gamma \in [0; 1]$ for the income gained by the agents. Hence, for agent i, the income is $H_i(y_i) = \lambda\ y_i$ and the goal function takes the form

$$f_i(y_i) = (1 - \gamma)\ \lambda\ y_i - c_i(y_i),\ i \in N. \qquad (1)$$

The deduction rate γ can mean income tax rate. Each agent chooses an action maximizing his or her goal function:

$$y_i(\gamma) = r_i\ \xi((1 - \gamma)\ \lambda),\ i \in N. \qquad (2)$$

The principal's goal function (as the total deduction obtained from the agents) makes up

$$\Phi(\gamma) = \gamma\ \lambda\ H\ \xi((1 - \gamma)\ \lambda), \qquad (3)$$

where $H = \sum_{i \in N} r_i$.

The principal strives for maximizing his or her goal function by choosing an appropriate deduction rate:

$$\Phi(\gamma) \to \max_{\gamma \in [0;1]}. \qquad (4)$$

For the Cobb-Douglas cost functions of the agents, i.e., $c_i(y_i) = \frac{1}{\alpha}\ (y_i)^{\alpha}\ (r_i)^{1-\alpha},\ \alpha \geq 1,\ i \in N$, one derives the following solution to the problem (4):

[18] Evidently, under the assumptions introduced below a uniform rate for all agents is optimal, see Section 2.7.

$$\gamma^*(\alpha) = 1 - 1/\alpha. \quad (5)$$

Hence, the optimal rate γ^* is an increasing function of α. The optimal value of the principal's goal function constitutes

$$\Phi_\gamma = \frac{\alpha - 1}{\alpha} \lambda H \xi(\lambda/\alpha),$$

i.e., $\Phi_\gamma = (\alpha - 1) H (\frac{\lambda}{\alpha})^{\alpha/(\alpha-1)}$, and the total agents' action is

$$Y_\gamma = H \xi(\lambda/\alpha) = H (\lambda/\alpha)^{1/(\alpha-1)}.$$

The gain of agent i is described by the formula

$$f_{i\gamma} = r_i (1 - 1/\alpha) (\lambda/\alpha)^{\alpha/(\alpha-1)}, \ i \in N,$$

while the sum of the goal functions of the system participants (the principal and all agents) is $W_\gamma = (\alpha^2 - 1)H(\lambda/\alpha)^{\alpha/(\alpha-1)}/\alpha$.

Centralized mechanism. Let us compare the obtained parameters with the corresponding values in another mechanism of economic motivation, *viz*., the *centralized scheme*. The principal "acquires" all income of the agents and compensates their costs incurred by the actions y_i provided that the plans x_i are fulfilled (a compensatory incentive scheme–see Sections 2.1, 2.3).

In this case, the principal's goal function takes the form

$$\Phi(x) = \lambda \sum_{i \in N} x_i - \sum_{i \in N} c_i(x_i). \quad (6)$$

By solving the problem $\Phi(x) \to \max_{\{x_i \geq 0\}}$, the principal evaluates the optimal plans:

$$x_i = r_i \xi(\lambda), \ i \in N. \quad (7)$$

For the Cobb-Douglas cost functions of the agents, the optimal value of the principal's goal function makes

$$\Phi_x = \lambda^{\alpha/(\alpha-1)} H (1 - 1/\alpha),$$

while the sum of agents' actions is $Y_x = H \xi(\lambda) = H \lambda^{1/(\alpha-1)}$.

The gain of agent i identically equals zero, since the principal exactly compensates the agent's costs and the sum of the goal functions of the system participants (the principal and all agents) is $W_x = \Phi_x$.

Compare the results of deduction mechanism and centralized mechanism:

- $\Phi_x / \Phi_\gamma = \alpha^{\frac{1}{\alpha-1}} \geq 1$ and decreases with respect to α;
- $Y_x / Y_\gamma = \alpha^{\frac{1}{\alpha-1}} \geq 1$ and decreases with respect to α;
- $W_x / W_\gamma = \alpha^{\frac{\alpha}{\alpha-1}} / (\alpha + 1) \geq 1$ and decreases with respect to α.

Therefore, if the agents have the Cobb-Douglas cost functions, the centralized mechanism of economic motivation is more beneficial for the whole organizational system than the deduction mechanism (indeed, the former ensures a greater total output and a higher total utility of all system participants than the latter).

The reservation "for the whole organizational system" seems essential, as in the centralized mechanism the profits (the goal function value) of the agents are zero–all available resources are accumulated by the "metasystem." Such scheme of interaction between the principal and agents may be inconvenient for them. Thus, let us analyze a generalized version of the centralized scheme known as the *profitability rate mechanism*. Here the reward of an agent (provided by a principal) not only compensates the costs under a fulfilled plan, but also includes a certain utility being proportional to the costs. The coefficient of proportionality is said to be the *profitability rate* (see Section 2.1, as well). The above centralized scheme corresponds to zero value of the profitability rate.

Profitability rate mechanism. Given a profitability rate $\rho \geq 0$, the principal's goal function is given by

$$\Phi_\rho(x) = \lambda \sum_{i \in N} x_i - (1 + \rho) \sum_{i \in N} c_i(x_i) . \qquad (8)$$

By solving the problem $\Phi_\rho(x) \to \max_{\{x_i \geq 0\}}$, the principal evaluates the optimal plans[19]:

$$x_{i\rho} = r_i \, \xi(\lambda / (1 + \rho)), \; i \in N. \qquad (9)$$

For the Cobb-Douglas cost functions of the agents, the optimal value of the principal's goal function makes up

$$\Phi_\rho = \lambda \, (\lambda / (1 + \rho))^{1 / (\alpha - 1)} H (1 - 1 / \alpha),$$

while the sum of the agents' actions is

$$Y_\rho = H \, \xi(\lambda / (1 + \rho)) = H \, (\lambda / (1 + \rho))^{1 / (\alpha - 1)}.$$

The gain of agent i constitutes $f_{i\rho} = \rho \, r_i \, (\lambda / (1 + \rho))^{\alpha / (\alpha - 1)} / \alpha$, and the sum of the goal functions of the system participants (the principal and all agents) is $W_\rho = \lambda H (\lambda / (1 + \rho))^{1 / (\alpha - 1)} (\alpha - 1 / (1 + \rho)) / \alpha$.

[19] Evidently, the principal has zero optimal profitability rate.

Again, we compare the derived results (note that, if ρ = 0, the formulas for the profitability rate mechanism yield the corresponding formulas for the centralized mechanism):

- $\Phi_x / \Phi_\rho = (1+\rho)^{\frac{1}{\alpha-1}} \geq 1$ and increases with respect to ρ;
- $Y_x / Y_\rho = (1+\rho)^{\frac{1}{\alpha-1}} \geq 1$ and increases with respect to ρ;
- $W_x / W_\rho = \dfrac{(1-\frac{1}{\alpha})(1+\rho)^{\frac{1}{\alpha-1}}}{1-\frac{1}{(1+\rho)\alpha}} \geq 1$ and increases with respect to ρ.

Interestingly, the maximal sum of the goal functions of the system participants (the principal and all agents) is attained by zero profitability rate (i.e., under absolute centralization)!

Now, compare the profitability rate mechanism with the deduction mechanism:

- $\Phi_\gamma / \Phi_\rho = (\frac{1+\rho}{\alpha})^{\frac{1}{\alpha-1}}$ and increases with respect to ρ;
- $Y_\gamma / Y_\rho = (\frac{1+\rho}{\alpha})^{\frac{1}{\alpha-1}}$ and increases with respect to ρ;
- $W_\gamma / W_\rho = \dfrac{(\alpha^2-1)}{\alpha^2 - \frac{\alpha}{(1+\rho)}} (\frac{1+\rho}{\alpha})^{\frac{1}{\alpha-1}}$ and increases with respect to ρ.

Thus, one draws the following conclusion. For the Cobb-Douglas cost functions of the agents, the profitability rate mechanism with $\rho = \alpha - 1$ is equivalent to the deduction mechanism.

This assertion follows from that under $\rho = \alpha - 1$ all (!) parameters of the profitability rate mechanism coincides with their counterparts in the deduction mechanism: $y_i(\gamma) = x_{i\rho}$, $i \in N$, $\Phi_\gamma = \Phi_\rho$, $Y_\gamma = Y_\rho$, $f_{i\gamma} = f_{i\rho}$, $i \in N$, $W_\gamma = W_\rho$.

To proceed, let us study the fourth mechanism of economic motivation–profit tax mechanism.

Profit tax mechanism. Suppose that the agent's income is interpreted as his or her goal function (the difference between the income and costs). Under a profit tax rate $\beta \in [0; 1]$, the goal function of agent i takes the form

$$f_{i\beta}(y_i) = (1-\beta)\,[\lambda\, y_i - c_i(y_i)],\ i \in N. \tag{10}$$

Accordingly, the principal's goal function is defined by

$$\Phi_\beta(y) = \beta\,[\lambda \sum_{i \in N} y_i - \sum_{i \in N} c_i(y_i)]. \quad (11)$$

In this case, the agents choose the same actions as in the centralized scheme; consequently, one obtains

$$y_{i\beta} = r_i\, \xi(\lambda),\ i \in N. \quad (12)$$

For the Cobb-Douglas cost functions of the agents, the optimal value of the principal's goal function constitutes[20]

$$\Phi_\beta = \beta\, \lambda^{\alpha/(\alpha-1)}\, H\,(1 - 1/\alpha),$$

while the sum of agents' actions makes up

$$Y_\beta = H\, \xi(\lambda) = H\, \lambda^{1/(\alpha-1)}.$$

The gain of agent i equals

$$f_{i\beta} = (1 - \beta)\, \lambda^{\alpha/(\alpha-1)}\, r_i\,(1 - 1/\alpha),$$

and the sum of the goal functions of the system participants (the principal and all agents) is

$$W_\beta = \lambda^{\alpha/(\alpha-1)}\, H\,(1 - 1/\alpha).$$

As usual, we compare the results:

- $\Phi_x / \Phi_\beta = 1/\beta \geq 1$ and increases with respect to β;
- $Y_x / Y_\beta = 1$;
- $W_x / W_\beta = 1$.

Therefore, the profit tax mechanism leads to the same sum of utilities and the same sum of equilibrium actions of the agents as its centralized counterpart. Yet, the principal's utility appears by β times smaller. Hence, the profit tax mechanism may be treated as the compromise mechanism, where the *compromise point* within the *domain of compromise* is defined by the profit tax rate (the share of allocating the system's profits between the principal and agents).

Compare the profit tax mechanism with the profitability rate mechanism:

- $\Phi_\beta / \Phi_\rho = \beta\ (1+\rho)^{\frac{1}{\alpha-1}}$;

[20] Obviously, for the principal the optimal value of the profit tax rate β is the unity (accordingly, the profit tax mechanism turns into the centralized mechanism).

- $Y_\beta / Y_\rho = (1+\rho)^{\frac{1}{\alpha-1}} \geq 1$;
- $W_\beta / W_\rho = \dfrac{(1-\frac{1}{\alpha})(1+\rho)^{\frac{1}{\alpha-1}}}{1-\frac{1}{(1+\rho)\alpha}} \geq 1$.

Finally, perform the comparison with the deduction mechanism:

- $\Phi_\beta / \Phi_\gamma = \beta\ \alpha^{\frac{1}{\alpha-1}}$;
- $Y_\beta / Y_\gamma = \alpha^{\frac{1}{\alpha-1}}$;
- $W_\beta / W_\gamma = \alpha^{\frac{\alpha}{\alpha-1}} / (\alpha + 1)$.

The analysis indicates that, the agents having the Cobb-Douglas cost functions, the profit tax mechanism possesses the following properties:

– for $\beta = 1 / \alpha^{\frac{1}{\alpha-1}}$ it is equivalent to the optimal deduction mechanism (according to the principal);

– for $\beta = 1 - 1 / \alpha^{\frac{\alpha}{\alpha-1}}$ it is equivalent to the optimal deduction mechanism (according to the agents);

– for $\beta = 1 / (1+\rho)^{\frac{1}{\alpha-1}}$ it is equivalent to the profitability rate mechanism (according to the principal);

– for $\beta = 1 - \rho / (\alpha - 1)\ (1+\rho)^{\frac{\alpha}{\alpha-1}}$ it is equivalent to the profitability rate mechanism (according to the agents).

Profit share mechanism. Here the principal gains the profits $H(y)$ as the result of agents' activity and pays a fixed share $\Psi \in [0; 1]$ of the profits to each agent (the same share is assigned to all agents–profit share mechanisma are unified). For agent i, the goal function takes the form

$$f_{i\Psi}(y) = \Psi H(y) - c_i(y_i),\ i \in N, \tag{13}$$

while the principal's goal function makes up

$$\Phi_\Psi(y) = (1 - n\ \Psi)\ H(y). \tag{14}$$

Under the profit share mechanism, the agents choose the actions:

$y_{i\Psi} = r_i\, \xi(\lambda\Psi),\ i \in N.$ (15)

Let the principal's profits be a linear function of agents' actions: $H(y) = \lambda \sum_{i \in N} y_i$. For the Cobb-Douglas cost functions of the agents, the principal's goal function has the value

$$\Phi_\Psi = (1 - n\,\Psi)\, H\, \lambda\, \xi(\lambda\Psi),$$

and the sum of agents' actions constitutes $Y_\Psi = H\, \xi(\lambda\Psi)$.

The gain of agent i equals

$$f_{i\Psi} = H\,[n\,\Psi\,\lambda\, \xi(\lambda\Psi) - \varphi(\lambda\Psi)],\ i \in N,$$

while the sum of the goal functions of the system participants (the principal and all agents) is

$$W_\Psi = H\,[\lambda\, \xi(\lambda\Psi) - \varphi(\lambda\Psi)].$$

In the case of quadratic cost functions of the agents, the optimal profit share (according to the principal) is $\Psi^* = 1/2\, n$.

Discussion. Thus, we have studied five mechanisms of economic motivation. In the aspect of efficiency (treated as the sum of utilities of all system participants and/or the sum of agents' actions), the best ones are the centralized mechanism and profit tax mechanism with any tax rate. Indeed, using the deduction mechanism or profitability rate mechanism yields a lower efficiency. Within the framework of the deduction mechanism, profitability rate mechanism or profit tax mechanism, one obtains different allocation of the principal's utility and agents' utility depending on the corresponding parameters (deduction rate, profitability rate and profit tax rate, respectively) as compared with the centralized mechanism (see the estimates above). In a specific situation, the derived formulas enable estimating the values of the parameters making the mechanisms equivalent. For instance, for quadratic cost functions ($\alpha = 2$) the optimal deduction rate (income tax rate) constitutes $\gamma^* = 0.5$. Full equivalence between the profitability rate mechanism and deduction mechanism is observed for $\rho^* = 1$.

Table 2.1. The parameters of the mechanisms of economic motivation: The case of quadratic cost functions of the agents

Name of a mechanism	Parameters			
	Φ	Y	W	Σf_i
Deduction mechanism	$\lambda^2 H/4$	$\lambda H/2$	$3\lambda^2 H/8$	$\lambda^2 H/8$
Centralized mechanism	$\lambda^2 H/2$	λH	$\lambda^2 H/2$	0
Profitability rate mechanism	$\lambda^2 H/(2(1+\rho))$	$\lambda H/(1+\rho)$	$\lambda^2 H(1+2\rho)/(2(1+\rho)^2)$	$\lambda^2 H\rho/(2(1+\rho)^2)$
Profit tax mechanism	$\beta\lambda^2 H/2$	λH	$\lambda^2 H/2$	$(1-\beta)\lambda^2 H/2$
Profit share mechanism	$\lambda^2 H/(4n)$	$\lambda H/(2n)$	$\lambda^2 H(2n-1)/(4n^2)$	$\lambda^2 H(n-1)/(4n^2)$

On the other hand, under $\beta^* = 0.5$ ($\beta^* = 0.75$) the profit tax mechanism appears equivalent to the both ones mentioned according to the principal (according to the agents, respectively). These conclusions are combined in Table 2.1.

A promising direction of further research lies in generalization of the obtained results to other classes of cost functions, as well as in the analysis of the mechanisms of economic motivation in some models of organizational systems with uncertain factors and interrelations of the agents.

Therefore, in the present chapter we have overviewed the basic results for the incentive problems. The following line of investigations seems to have prospects here: deriving analytical solutions to the problems of optimal incentive mechanism design in multilevel multi-agent dynamical systems with uncertain parameters and distributed control. From practical considerations, the results of theoretical study can be used to develop and adjust software tools of personnel management.