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**REGULARITIES
OF ITERATIVE LEARNING**

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The research is devoted to the study of common for systems of animate and inanimate nature - a person, a group of people, animals, artificial systems - of the quantitative laws of iterative learning (understood as repeated repetition of actions, trials, attempts, etc., by a trained system to achieve fixed goal under constant external conditions). The main research method is mathematical modeling.

The book (which is a translation from first edition, published in 1998 in Russian) is aimed at specialists in pedagogy, psychology and physiology of humans and animals, control theory, as well as undergraduate and graduate students of relevant specialties.

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Introduction

Iterative learning, as learning in strictly repeating conditions, is one of the simplest types of learning, which takes place in a wide class of phenomena: the formation of diverse skills, the assimilation of information by humans, the teaching of animals (the development of conditioned reflexes), and the training of technical and cybernetic systems. Various aspects of iterative learning are studied in pedagogy, psychology and physiology of humans and animals, in control theory and in other sciences.

The present work is devoted to the description of mathematical models of iterative learning and pursues the following goals:

- firstly, to give a fairly complete, although certainly not exhaustive, analytical review of the currently existing models of iterative learning proposed by different authors in different years (more than thirty of such models are considered below), including the author of this paper.

- secondly, on the basis of the analysis of the described models, try to identify and explain the most important general laws and mechanisms of iterative learning, as well as to determine the possibilities of mathematical modeling as a method of studying iterative learning.

1. Problems of iterative learning modelling

The present work is devoted to the description and study of mathematical models of iterative learning. Therefore, first of all, it is necessary to determine what is meant by “model” and “iterative learning”.

We will use the term “*model*” in its broadest sense as “an analogue of a certain fragment of natural or social reality, ... a deputy of the original in cognition and practice”, a mathematical (abstract) model - “interpretation of systems of logical and mathematical positions” (M.: Soviet Encyclopedia, 1983. p. 382).

Learning in the general case is “the process and result of the acquisition of individual experience” (Brief Psychological Dictionary, Moscow: 1985. p. 201).

We will consider in detail only one of the types of science, namely *iterative learning* (iterative from lat. Iterativus - repeated) - sequential repetition by the system (living or non-living - technical or cybernetic) of actions, tests, attempts, etc. to achieve a fixed goal under constant external conditions. Iterative learning (IL) underlies the formation of human skills, conditioned reflexes in animals, the training of many technical (materialized) and cybernetic (abstract-logical) systems and is the subject of research in educational and engineering psychology, psychophysiology, pedagogy, control theory, etc. IL refers to relatively simple types of learning and its study expands the understanding of the mechanisms of learning in general.

The constancy of external conditions allows for a quantitative description of IL in the form of graphs - *learning curves* (LC), which are a dependence of the learning level criterion on time or on the number of repetitions (iterations).

Numerous experimental data indicate that the most important general pattern of iterative learning in living systems (humans, groups of people, animals) and inanimate systems (pattern recognition systems, probabilistic automata with variable structure, neural networks, etc.) is the slow-asymptotic nature of learning curves: they are monotonous, the rate of change of the learning level criterion decreases with time, and the curve itself asymptotically tends to a certain limit. In most cases, iterative learning curves are approximated by exponential curves (see Section 2 for more details).

We will be interested mainly in the following question - what is the reason for the general regularity for iteratively learned systems of the most diverse nature, which consists in the slow-asymptotic nature of LC?

There are various approaches to obtaining an answer to this question: the study of experimental data (phenomenological description); analysis of the psychophysiological or technical characteristics of the trained systems, their structure, the principles of interaction of their constituent elements; creation and research of mathematical, simulation, and other models of IL, etc. We will try to consider the general laws of IL by examining its models.

Thus, the object of study in this work is iterative learning, and the subject of the study are its quantitative laws common to systems of animate and inanimate nature, and the main method of research is mathematical modeling. The aim of the work is a theoretical justification and explanation of the general laws of IL and, accordingly, tasks are: analysis of the known and the construction of a number of new mathematical models of iterative learning; establishing the adequacy of models to real systems; consideration of the possibility of explaining the known and predicting new properties of iteratively taught systems and the process of IL through modeling.

There are two methods of constructing models in general, and, accordingly, they can be used to build models of iterative learning - direct and reverse.

When using the direct method, certain assumptions are made about the functions, composition and structure of the learning system and the mechanisms of interaction of its constituent elements. Further, on the basis of the assumptions introduced and the laws "incorporated" in the model, the model behavior is investigated and the correspondence of the model behavior to the modeled system is analyzed. The explanatory and prognostic properties of the model are determined by the generality of the hypotheses used in its creation. It is clear that, despite the identical behavior of the model and the simulated system, the laws of interaction of their elements, and their structure, may have nothing in common. Nevertheless, if the hypothesis that the model is "arranged" in the same way as the training system is justified, then the analysis of the model allows one to transfer a number of results and recommendations on the organization of its more expedient functioning to the simulated system itself. For example, sometimes recommendations on the possibilities of increasing the effectiveness of learning in the framework of a particular model can be used to select the optimal organization of the real educational process (reducing the time

spent on training, reducing costs, increasing the productivity of the actions of the learning system, etc.).

The second, inverse method of constructing models is to search for those initial assumptions and assumptions that lead to the required properties of the model. For example, if the trajectory of the motion of a system and its structure are known, then sometimes, in accordance with the inverse method, one can find the class of laws of interaction of elements of the system with each other and with the environment, leading to the observed behavior. In this case, the "internal device" of the model can be very different from the "device" of the simulated system. For example, if various assumptions about the laws of interaction lead to the same result, then, without additional information, it is impossible to unequivocally say which of the equivalent models correspond to a real system.

The division into direct and inverse methods for constructing models is rather arbitrary - most of the currently known models of IL use to some extent both of these approaches. The process of constructing a model (mathematical, simulation, etc.) is, as a rule, iterative in nature. First, the researcher makes assumptions about the structure of the model and the laws of interaction of elements, consistent with the available information about the simulated system (using the direct method). Then, the model's behavior is compared with the original's behavior and, based on this comparison, changes are made to the accepted hypotheses and assumptions, assumptions are "minimized" (using the inverse method), after which the model's behavior is again examined, etc. Conventionally, we can assume that the successful use of the direct method leads to the finding of sufficient conditions (one degree or another of generality) of adequacy. The purpose of the inverse method is to search for the necessary conditions of adequacy. Therefore, it should be recognized that the inverse method is more constructive; since the model constructed using it allows one to draw more reasonable conclusions about the internal structure, mechanisms and processes in real simulated systems. At the same time, it is clear that in doing so the researcher will undoubtedly encounter great difficulties.

From the above it follows that it is possible to build many direct models of the same real system or process. However, it is very rarely possible to create a model that is adequate to the original not only in behavior, but also in structure, functioning mechanisms, etc. Those rare cases in which the structure and properties of the model can be unambiguously (necessarily) derived and identified by information about the simulated system

should be recognized as successful exceptions to the general pattern. When modeling most complex (especially biological and socio-economic) systems, including when modeling iterative learning, we should talk about a harmonious combination of direct and inverse methods.

Here is one of the current views on the possibility of creating a general model of iterative learning (E. Guthrie): "For many years, researchers have been inspired by the hope of opening the learning curve. There is a general agreement that the curve changes more quickly after the start of the exercise, as the exercise continues this speed gradually decreases until the physiological limit set by the nature of the learner is reached Of course, there is no ideal standard learning curve or forgetting curve from previous experience in practicing action components and already formed skills ... In other words, there is no general learning curve. " [30, p. 179].

The above view of E. Guthrie is perhaps too pessimistic. It all depends on what is meant by the "generality" of the model. If the "general" model is a universal model explaining and generalizing all known models and a priori able to explain all the possible effects that are still unknown today that are observed during iterative learning, and adequate at any level of detail to consider an arbitrary system, then perhaps the possibility of creating such Models today seem problematic.

At present, a large number of studies are known that explain, under certain assumptions and assumptions, the regularities of IL for specific systems (it is interesting to note that over the past few decades there has been a decline in the intensity of studies of general models of iterative learning; therefore, it is not surprising that most of the works cited below in the list of literature relate to 60-70 years - the period of rapid development of cybernetics). However, from our point of view, most of the existing models do not have sufficient generality. Therefore, it makes sense to talk about creating the most general IN model (or a complex of such models), using the minimum assumptions and assumptions about the structure of the learning system, the properties of its constituent elements and the nature of their interaction, as well as highlighting those general assumptions, hypotheses, and so on, which are used in the well-known and should be used in any mathematical models of IL.

The wildlife systems of interest to us are large and complex, both in terms of the number of elements making up them and in terms of the variety of connections between them [7, 11, 35, 77, 78]. In technical systems and models of living systems, a researcher can artificially limit complexity

by making the system amenable to analysis. For example, at the moment, the properties of only individual elements of these living systems — neurons, synergies, etc., can be approximately described, their characteristics are measured with varying degrees of detail, and the relationships between them are described. However, sadly enough, a sufficiently complete answer to the question has not yet been received: how the brain functions, and how the properties of individual neurons lead to those properties of their groups, individual subsystems and the brain as a whole, which we observe.

The limited modern scientific knowledge in understanding the mechanisms of functioning of biological and social systems complicates the task of modeling iterative learning even more - if we do not have a clear idea of the properties of a real system, then it is unclear what is meant by the adequacy of the model and system at the level of "internal structure". This is probably why most of the IL models are phenomenological in nature, describing the aggregated dynamics of the effective characteristics of learning, but not "looking inside" the simulated system.

We will try to formulate, in general terms, what kind of conclusion we would like to receive in this paper. One can hardly hope that for iteratively learned systems it will be possible (someday) to obtain a universal law at the level of the basic laws of nature or to prove an appropriate general formal result, since for this it is necessary to introduce a system of axioms - postulates, the obviousness of which may turn out to be (and it turns out in existing models) far from indisputable. Therefore, it is desirable to formulate and justify a pattern that, firstly, explains the experimentally observed behavior of iteratively learned systems, and, secondly, would have the greatest possible generality (i.e., would be applicable to the widest possible class of learned systems and would require the introduction of minimal assumptions and assumptions).

Note that most of the known and used principles and laws of the functioning of biosystems are precisely the nature of laws or hypotheses. To illustrate this statement, without pretending to be a complete description, we briefly list some well-known principles of the functioning of biological systems.

1. **The principle of least action.** When some change occurs in nature, the amount of action required for this change is the smallest possible [4].

2. **Law of sustainable imbalance.** All living and only living systems are never in equilibrium and, due to free energy, perform constant work

against the equilibrium required by the laws of physics and chemistry under appropriate external conditions [10].

3. **The principle of the simplest construction.** That particular structure or construction of a living system that we really find in nature is the simplest possible structure or structure capable of performing a given function or structure of functions [59].

4. **The principle of feedback** [78] (see also the *principle of the functional system* of P.K. Anokhin [2, 3]). Here it is appropriate to mention the principle of anticipatory reflection of reality - a complex adaptive system does not respond to external influences as a whole, but according to the "first link of a repeated series of external influences many times". A necessary condition for such anticipatory reflection is the consistency and repeatability of external phenomena (in the case of iterative learning - the constancy of external conditions and learning objectives) [3, 45].

5. **The principle of least interaction.** Nerve centers strive to achieve a situation in which afferentation (from the Latin afferentis - bringing, that is, information and control flows and signals transmitted in the central nervous system) is the smallest. Or, in other words, the system expediently works in some external environment, if it seeks to minimize interaction with the environment [73].

6. **The principle of the probabilistic functioning of the brain.** Each of the neurons does not have an independent function, that is, a priori is not responsible for solving a specific problem, the distribution of which occurs in a rather random way [4, 36].

7. **The principle of hierarchical organization,** in particular, information processing by the brain. Achieving the full goal is tantamount to achieving a set of subgoals [1, 6, 12, 76]. "... in each complex system, control and working floors can be distinguished" [1, p. 81].

8. **The principle of adequacy.** The complexity of the control system (the dynamics of its changes) should be adequate to the complexity (rate of change) of the controlled processes. In other words, the "throughput" of the regulator sets the absolute control limit, however great the possibilities of the controlled system [7].

9. **The principle of probabilistic forecasting** when constructing the actions. The world is reflected in the form of two models - the model of the required future (probabilistic forecasting based on previous experience) and the accomplished-model (unambiguously reflects the observed reality) [12, 45]. This approach is fully consistent with the following definition of

learning: "Learning a system is that, in accordance with previous successes and failures (experience), it improves the internal model of the outside world" [66, p. 228].

10. The principle of selection of the necessary degrees of freedom.

At the beginning of training, a greater number of degrees of freedom of the learning system is involved than is necessary to achieve the learning objectives [12]. In the learning process, the number of "participating" variables decreases - non-essential variables are "disabled".

11. The principle determinism destruction. In order to achieve a qualitatively new state and increase the level of organization of the system, it is necessary to destroy (rebuild) the existing, determined in previous experience, deterministic structure of the connections of system elements [5, 24].

12. The principle of necessary diversity. This principle is quite close in meaning to the principle of adequacy: to solve the problem facing it, the system must have the corresponding diversity (states, functions, capabilities, etc.), that is, the system must be adequate to the task in the sense of diversity (complexity) [7].

13. The principle of natural selection. In systems that have become effective as a result of natural selection, the diversity of mechanisms and the throughput of data transmission channels will not significantly exceed the minimum value required for this [6].

14. The principle of deterministic representation. When modeling decisions by an individual, it is assumed that his ideas about reality do not contain random variables and uncertain factors (the consequences of decisions depend on strictly defined rules) [37].

15. The principle of complementarity (incompatibility) (N. Bohr, L. Zadeh). High accuracy of the description of a certain system is incompatible with its great complexity. Sometimes this principle is understood more simply -- the real complexity of the system and the accuracy of its description in the analysis are inversely proportional to a first approximation.

16. The principle of monotonicity ("do not miss what has been achieved"). In the processes of learning, self-organization, adaptation, etc. the system, on average, does not move away from the already achieved (current) positive result (equilibrium position, learning goals, etc.) [7].

At first glance, the above principles of the functioning of biosystems can be conditionally divided by approaches into natural-scientific ap-

proaches: 1, 2, 5, 8, 15; empirical approaches: 4, 6, 10, 11, 14, 16 and intuitive approaches: 3, 7, 9, 12, 13. Physical approaches ("laws") reflect the general laws, limitations, and possibilities of biosystems imposed by the laws of nature. Empirical principles are usually formulated on the basis of the analysis of experimental data, the results of experiments and observations, and are more local in nature than natural sciences. Finally, intuitive laws and principles (which in theory should not contradict the natural sciences to be consistent with the empirical ones) are the least formal and universal in nature, based on intuitive ideas and common sense.

In fact, a closer examination reveals that all the above "natural science" principles are more empirical and / or intuitive. For example, the principle of least action, which would seem to be a classical physical law, is formulated for mechanical systems (there are analogues in optics and other branches of physics). Its unadapted use in the study of biological systems, generally speaking, is not entirely correct and justified. That is, the assertion that biosystems satisfy the principle of least action is just a hypothesis introduced by researchers and not supported today by correct justifications.

Thus, the well-known principles (and laws) of the functioning of biosystems fit into one of the following formulations: regularity - "if the system has some (specific) internal structure, then it behaves in an appropriate (certain) way" or: hypothesis - "if the system behaves in a certain (specific) way, then it most likely has a corresponding (defined) internal structure." Addition - "most likely" is essential: the first type of statements establishes sufficient conditions for the implementation of the observed behavior (see the description of the direct and inverse methods above) and can be partially or completely confirmed experimentally; statements of the second type are in the nature of hypotheses - "necessary" conditions (in most cases hypothetical and unproven and fulfilling an explanatory function) superimposed on the structure and properties of the system based on its observed behavior.

Therefore, based on an analysis of the iterative learning models studied in this work, it is desirable to formulate a regularity of the form: "if the system being taught has the following properties ... and functions under the following conditions ... then the learning curves will be exponential", and, in fact, an explanation of the laws IL is a hypothesis of the form: "if the learning curves of some iteratively taught system are exponential, then the

system most likely has the following properties ... and functions under the following conditions ...".

So, we see that the above principles of the functioning of biosystems are either empirical or intuitive. Accordingly, we can distinguish two areas of iterative learning research and two ways of formulating and explaining its mechanisms. The first method is the analysis of experimental data. A review of the work on the experimental results of studying IL (and there are thousands of such works!) Is beyond the scope of this study, although it can be argued that in most cases the experimental dependences are approximated by delayed asymptotic curves [8, 16, 20, 22, 23, 29, 30, 33, 47, 48, 67, 68, 69, 70]. The second approach - the creation and analysis of models - is discussed below. An analysis of well-known models, as well as the synthesis and study of new mathematical models of IL, as will be seen from the subsequent discussion, will allow us to generalize approaches to modeling iterative learning and explain some patterns of not only IN, but also control processes, self-organization and adaptation for a very wide class of complex systems.

2. Learning curves: quantitative description and qualitative analysis

When studying any system, including a biological one, conducting a physical experiment, studying a black box, etc., it is possible to establish causal and quantitative relationships between input and output variables only if the output signal changes (system response) caused by a change in one of the input signals. If two or more input variables have changed simultaneously, then in the general case it is impossible to distinguish what affect each of the inputs had on the observed change in the output variable.

There are two aspects of learning. The first aspect is the productive one - when learning, the system must achieve the desired result - the quality of the actions with acceptable costs of time, energy, etc. The second aspect is the procedural one: adaptation, adaptation of the system being taught to a certain type of action during the exercise, etc. Accordingly, distinguish the effective characteristics of iterative learning and characteristics of adaptation [48]. In this work, we are talking about the effective characteristics of learning (adaptation characteristics often have completely different dynamics).

In the case of iterative learning, it can be considered that its output characteristics are affected by two input variables - information about the value of the output variable and environmental parameters - external conditions. If both values of the input variables were changed at some step, then the learning results at this step and at the previous one would be simply incomparable - it would not be possible to say why this value of the output variable was realized: because the learning system behaved accordingly, or because the conditions of its functioning have changed. Therefore, the constancy of external conditions is an essential characteristic of IL. For comparability of the results of learning at different points in time (using a quantitative description), even under constant external conditions, the constancy of the goal of learning is also important.

The criterion of the level of learning is usually taken as the main effective characteristic of IL. When learning real systems, the following characteristics can serve as a criterion for the level of learning [47, 48]:

- temporary (time taken to complete the action, operation, reaction time, time taken to correct the error, etc.);
- high-speed (labor productivity, reaction rate, movement, etc. - the reciprocal of time);

- accuracy (error value in terms of physical quantities (millimeters, angles, etc.), the number of errors, the probability of error, the probability of an exact reaction, action, etc.);
- informational (the amount of memorized material, processed information, the amount of perception, etc.).

Since the models of iterative learning are mainly considered below, for the sake of generality of presentation, we will call the productive characterization of learning of interest to us the mismatch. Indeed, in all of the cases listed above, we have either a function of error (mismatch) or a characteristic of the “learnedness” of the system, which can be reduced to some error function. For example, the execution time of an action can be interpreted as a mismatch, if by the latter we mean the difference between the current value of the time of the execution of the action and the minimum possible.

As noted above, iterative learning is typically characterized by slow-asymptotic learning curves approximated by exponential curves. In general, the exponential curve is described by equation

$$(2.1) \ x(t) = x^\infty + (x^0 - x^\infty) e^{-\gamma t}, \ t > 0,$$

or

$$x_n = x^\infty + (x^0 - x^\infty) e^{-\gamma n}, \ n = 0, 1, 2, \dots, m,$$

where t is the time of learning, n is the number of iterations (tests, attempts) from the moment of learning (it is assumed that learning begins at time zero), $x(t)$ (x_n) is the value of the mismatch at time t (at the n -th iterations), x^0 is the initial value of the mismatch (corresponding to the moment of the beginning of learning), x^∞ is the "final" value of the mismatch (the value to which the SC asymptotically tends; as a rule, in biological systems this quantity is considered as the physiological limit of learning), γ is some non-negative constant, defining speed changes and LC- learning rate (γ has dimension inverse time or number of iterations). Sketches of the graphs of the curves (2.1) are shown in Figures 2.1.a and 2.1.b.

Depending on the ratio of the initial and final values of the mismatch, expression (2.1) describes both increasing and decreasing LC: for $x^0 > x^\infty$ the curve will be increasing, and for $x^0 < x^\infty$ – decreasing. The quantitative characteristics of learning (x^0 , x^∞ , γ) depend on many factors: the complexity and properties of the learning system, the external environment, the teaching methods used, etc.

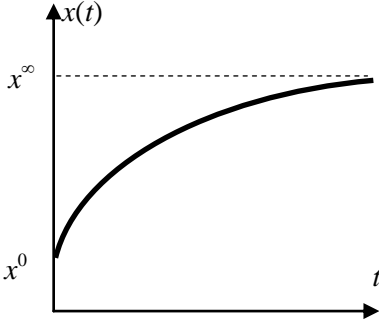


Fig. 2.1a.
Increasing LC ($x^\infty > x^0$)

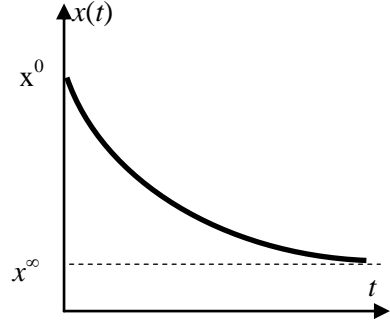


Fig. 2.1a.
Decreasing LC ($x^\infty < x^0$)

We will be interested mainly in the qualitative type of the LC, therefore, in most cases, for simplicity, we will use the following more particular dependencies:

$$(2.2) \quad x(t) = e^{-\gamma t}$$

$$(2.3) \quad x(t) = 1 - e^{-\gamma t}.$$

If we are talking about the magnitude of the error, then in accordance with (2.2), the error decreases monotonously. If x is interpreted, for example, as a “level of knowledge,” then it, in accordance with (2.3), increases monotonously. Obviously, (2.2) and (2.3) can be obtained from the general dependence (2.1) using the linear transformation:

$$x_{(2.2)} = \frac{x_{(2.1)} - x^\infty}{x^0 - x^\infty}, \quad x_{(2.3)} = \frac{x^0 - x_{(2.1)}}{x^0 - x^\infty}.$$

Therefore, speaking of the learning curve, we will mean a family of curves equivalent up to a linear transformation. A characteristic of the family is a value that is the same for all LC from the equivalence class under consideration, in this case the learning rate will be. Sketches of the graphs (2.2) and (2.3) are shown in Figures 2.2.a and 2.2.b, respectively.

It should be noted that to date, a significant number of different approaches to the approximation of learning curves and exponential LC of the form (2.1) are known, although they are the most common, but not the only ones. Without pretending to be a complete description, we list some known dependences (see reviews of LC in [36, 38, 47, 50, 57, 64, 75, 79]).

For the first time, the idea of using inductive reasoning in pedagogy and psychology was put forward in 1860 by G. Fechner, who proposed,

having collected a sufficiently large number of experimental data, to approximate them with the most suitable analytical function. Since then, both psychology and pedagogy in the quantitative description of phenomena and processes in most cases follow this path [79].

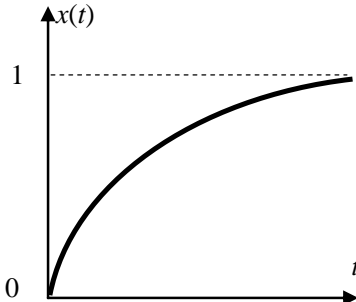


Fig 2.2a

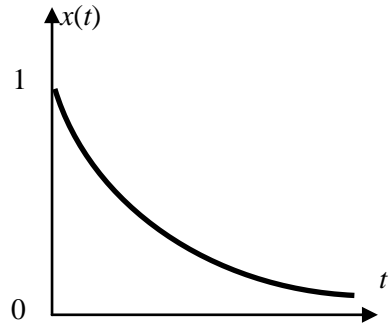


Fig 2.2a

Normalized ILC (increasing - a, decreasing - b)

The approximation of the “forgetting curves” proposed by H. Ebbinghaus (1885 — apparently the first quantitative descriptions of LC) was based on an exponential function, although quite different from (2.1) [20]. The explanation of this difference is quite simple - a person has “short-term” and “long-term” memory, characterized by different times of memorization and storage of information.

Using the assumption that there is an analogy between the learning process and a monomolecular chemical reaction (see model 5.2 below) leads to an exponential dependence: $x(t) = \alpha + \beta e^{-\gamma t}$, where α, β are some constants. By analogy with a monomolecular autocatalytic reaction or by using analogies with the chemical law of the acting masses [22]: $x(t) = \alpha e^{\gamma t} / (\beta + e^{\gamma t})$.

Thurstone L., on the basis of a generalization of the experimental material Lashley K. (training rats to find the path in the maze), proposed to approximate the accumulated error (i.e., the total error starting from time zero or the first iteration) by the following formula:

$$(2.4) \quad x(n) = \alpha n / (b + n),$$

where n – number of exercises, α, β are some positive constants [68].

Предложенное Н. Gulliksen в [29] the empirical LC equation for accumulated errors in the passage to the limit (a sufficiently low learning speed and reinforcement force) goes over into (2.1), i.e., the LC is approximated by the exponent.

The averaged LC obtained by R. Atkinson and colleagues [8] in accordance with the theory of stimulus selection is close to exponential function.

It should be noted that in many works it was pointed out that it is necessary to study the learning curves averaged (over the subjects — their group, or over time), since individual SCs have, as a rule, a significant spread ("... smooth LCs are the result of the averaging process .. ." [22, p. 392]) [30, 34].

In [60], to describe the quantitative relationship of factors of reinforcement, non-reinforcement, and conditioned reaction in experiments on the formation of conditioned reflexes, a formula of the form (2.4) was proposed (for the dependence of the level of formation of a conditioned reflex on the number of reinforcements of a conditioned stimulus).

Various researchers used exponential functions, hyperbolas, parabolas, etc. [57] to approximate the experimental learning curves by various researchers. LC differed with increasing, decreasing, and constant growth [23]. Postponing the discussion of the diversity of approaches, we note that when comparing various IL descriptions, it is necessary, first of all, to pay attention to whether this learning is iterative, what indicators are analyzed as characteristics of the learning's effectiveness and in what scale these indicators are measured.

Since iterative learning is one of the special cases of learning, then, in addition to exponential curves corresponding to iterative learning, there are other types of LC, including logistic LC.

Logistic learning curves are approximated by dependency
 (2.5) $x(t) = x^0 x^\infty / (x^0 + (x^\infty - x^0) e^{-\gamma t})$,
 and depending on the ratio of the initial and final values, the mismatches can be either increasing or decreasing [67]. A sketch of the graph of a normalized increasing logistic curve is shown in Figure 2.3.

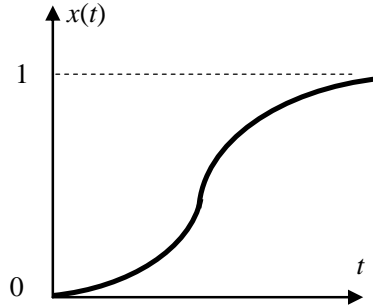


Fig. 2.3. Logistic LC

With relatively complex types of learning, the LC can have a plateau, the presence of which is explained by the hidden search by the learning system for new ways to improve the ways of performing actions, preparing for the transition to a qualitatively new way of mastering the activity, for a new strategy [15, 30]. In figure 2.4. a fairly common type of IL with an intermediate plateau is given: two successive exponents correspond to the development of two different action strategies.

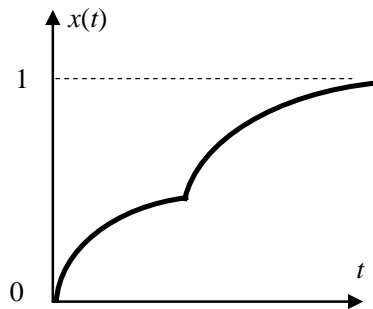


Fig. 2.4. LC with plateau

Several initial samples can be spent searching for the most appropriate tactics of behavior, which leads to the presence of an initial plateau on the logistic curve [48]. In complex learning processes, in accordance with [16], three stages can be distinguished. The first stage is characterized by the selection of a large number of stimuli "significant" stimuli. This stage can

be considered as the formation of the initial field of events. The second stage is characterized by the development of the correct behavior, determined by the selected system of events (the iterative learning itself is precisely the second stage). The third stage is characterized by a relatively stationary level of training.

And finally, when using dichotomous scales (when some critical level of error is arbitrarily set; if during the course of the action the error is less than the critical value, then the action is considered to be performed correctly) or the learning level is chosen as a criterion for time, accuracy the execution of the action and the amount of processed information of the quantities, that is, when using the divisor transformation (reaction rate, labor productivity, etc. - as quantities inverse to time, etc.), my Logistic curves can be found. In this case, their appearance is somewhat unnatural and can be eliminated by choosing the appropriate scale and units. It can be shown that by constructing the inverse for the exponential curve or by discretizing the scale, we can obtain the logistic LC [47, 55, 67].

Learning curves corresponding to non-productive characteristics of learning, including iterative learning, that is, adaptation characteristics, can be combinations of exponential and logistic LC, step-like, or any other, including nonmonotonic curves. Such LC that characterize the internal structure of actions, including, for example, during the formation of various skills in humans and animals, can be observed in complex types of learning: with a consistent deep restructuring of the structure of the skill, organization of phased development of individual components of actions, etc. [48]. In the future, we will consider learning curves that correspond only to the productive characteristics of iterative learning.

The regularity of iterative learning (as the simplest type of learning in general), consisting in a slow-asymptotic form of learning curves corresponding to the effective characteristics of IL, indicates the presence of common learning mechanisms in living objects - humans, groups of people, animals and their artificial analogues - technical and cybernetic systems. Without giving detailed experimental data - they are contained in the cited literature, below we will try, analyzing the mathematical models of IN, to find out what lies at the basis of these general laws.

3. Classification of iterative learning models for men, animals and artificial systems

Most models of iterative learning are based on analogies with phenomena and processes occurring in various systems of animate or inanimate nature. Therefore, it is natural to put in the basis of classification the type of process or phenomenon, the analogy with which is used.

Figure 3.1 shows the proposed classification system for iterative learning models.

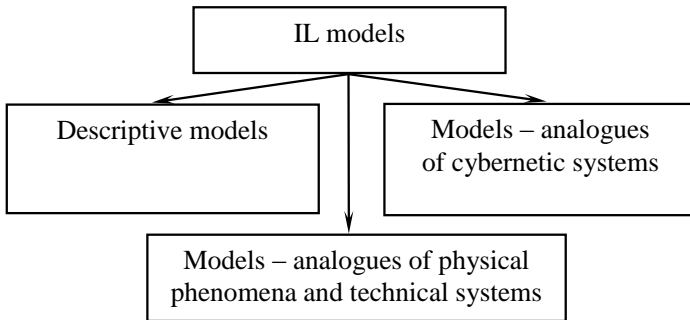


Fig. 3.1. Classification of IL models

In descriptive models (axiomatic and intuitive), certain assumptions are introduced (postulated) about the relationship of variables and system parameters, and these assumptions and the model of the learning system, as a rule, are quite abstract and do not appeal to real counterparts (in intuitive models they are based on intuition and common sense). This class of models is considered in Section 4.

Section 5 is devoted to the description of IL models using analogies with the provisions of physical phenomena and the principles of functioning of technical systems. Their subclass - information-theoretic models - is placed in a separate section due to its specificity and diversity (section 6).

Models using analogies of cybernetic systems - section 7 and collective behavior models - section 8 are interesting in that they are artificial, rather abstract models, and those systems, by analogy with which they are built, are often, in turn, models of some real systems (models - analogies of models).

Since the analogies used are quite diverse, we will try to present on the most generalized level, specifying the meanings of certain terms only when it will be necessary to prevent ambiguity in understanding. We give the general structure for describing the mathematical model of iterative learning.

Suppose that a learning system (hereinafter referred to simply as a “system”) consists of n , generally interacting, elements ($n > 1$), each of which is described by some scalar parameter $x_i(t)$, which depends on time, which we will hereinafter arbitrarily call the mismatch of the i -th element. The mismatch of the system $x(t)$ somehow depends on the mismatch of its constituent elements:

$$x(t) = F(x_1(t), x_2(t), \dots, x_n(t)).$$

This description is common to most models, which are also assumptions about the interaction of elements (functions $F(\cdot)$).

The entire presentation of the models below is constructed according to the following scheme (some of the stages can be omitted or differ in meaningful interpretations of the terms "system", "element", "parameter", "mismatch", etc., but combined with others):

- model description (**D**) - description language, subject area, factors and variables;

- hypothesis (**H**) - assumptions about the relationship of variables, interaction mechanisms, etc.;

- formal (logical, algebraic, etc.) transformations (**T**);

- conclusion (**C**) (the conclusion from the analysis of most of the models given below is “the mismatch is described by a dependence of the following form ...”, and this dependence is, as a rule, exponential);

- analysis of the model (**A**) - discussion of the hypothesis, assumptions, their validity, the study of factors affecting the speed of learning, etc.

The speed of learning, in the general case, depends on all the parameters of the model: the number of elements, relationships and the laws of their interaction. Knowledge of the type of this dependence seems quite important, since the study of the parameters that determine the speed of learning is essential for finding ways to increase the effectiveness of learning and, in the first place, the speed of learning. Indeed, knowing the dependence of the learning rate on the parameters of the model, it is possible to propose measures leading to a corresponding change in these parameters, and, therefore, the required change (most often to increase) in the learning speed.

A description of models that do not belong to the author of this work is accompanied by links to relevant sources (see the list of references). In such models, the statement, with the exception of stage A — analysis, follows the original — the work of the authors of the models.

It should be recognized that in order to ensure uniformity and simplicity of presentation, the author had to allow a number of “liberties” that could cause fair objections from the mathematician reader. So, for example, difference and differential equations are sometimes identified and statements about "correspondence" between their solutions are given. In the latter case, in models with discrete time, by an exponential “curve” we mean a sequence of values of the learning level criterion, the elements of which constitute a geometric progression.

The completion of each model description is indicated by "•".

4. Descriptive models: axiomatics and intuition

By descriptive we will understand models of iterative learning, in which there are no obvious analogies with the principles of the design and functioning of various systems, and the exponential form of SC is obtained as a result of the introduction of sufficiently abstract and unreasonable assumptions regarding the laws and rules of interaction of elements of the student systems (in axiomatic models sometimes it is directly postulated that the learning curve is described by an exponent - expression (2.1)). In most cases, in descriptive models, the assumptions introduced are based on intuition and appeal to common sense, and conclusions from an analysis of the dynamics of LC often underlie higher-level models [79].

Model 4.1.

D. Change in system mismatch over time.

H(C, T). The rate of change of the mismatch is proportional to its current value, and the coefficient of proportionality is independent of time:

$$(4.1) \quad \frac{dx(t)}{dt} = -\gamma x(t).$$

The conclusion is obvious - the exponent is the solution to this differential equation - expression (2.1).

A. A significant part of axiomatic models in one way or another suggests proportionality between the change in the mismatch per unit time and its current value. It is clear that with a constant coefficient of proportionality, such an assumption immediately leads to an exponential type of SC, and to increase the learning speed it is necessary to increase the coefficient γ , which in the future will be interpreted in various models as the amount of information processed by the trained system per unit time, the communication channel bandwidth, objectively existing restriction on the rate of change of parameters of elements, etc.

Similar constructions (although with somewhat more artificial initial hypotheses) are given in [23]. In the model with discrete time, if:

$$x_n - x_{n-1} = -\alpha x_n,$$

then

$$x_n = (1 - \alpha)^n x_0, \quad n = 1, 2, \dots,$$

and the learning rate decreases with increasing α ($\alpha \in (0; 1)$). If $x_n = \beta x_{n-1}$, then $x_n = \beta^n x_0$, $n = 1, 2, \dots$, and the learning rate increases with increasing β ($\beta \in (0; 1)$). •

Model 4.2. [16, 22, 34].

D. Mismatch is the probability of a correct reaction (for example, in the famous experiment "rat in a labyrinth") [8, 16]. The dependence of the mismatch on the number of repetitions is investigated. If the probability of a correct reaction is p (the probability of a wrong reaction is $(1 - p)$, respectively), then it can increase by no more than $(1 - p)$, become equal to one, and decrease by no more than p , and become equal to zero.

H. At each step, the increase in the mismatch is proportional to the possible increment, and the decrease is proportional to the possible decrease. The difference equation for the probability of a correct reaction has the form:

$$(4.2) \quad x_n = x_{n-1} + \alpha_n (1 - x_n) - \beta_n x_{n-1}, \quad n = 1, 2, \dots,$$

where $\alpha_n, \beta_n > 0$.

T(C). Under initial state x_0 and constant coefficients α ($\alpha_n = \alpha$), and β ($\beta_n = \beta$) obtain

$$x_n = x_0 (1 - \alpha - \beta)^n + \alpha \sum_{k=0}^{n-1} (1 - \alpha - \beta)^k.$$

The continuous "analog" of this solution has the form

$$x(t) = x^\infty + (x^0 - x^\infty) e^{-(\alpha + \beta)t},$$

where $x^\infty = \alpha / (\alpha + \beta)$.

A. Compared with the previous model, a complication is introduced in the model considered here - the possibility of both increasing and decreasing the mismatch (cf. (4.1) and (4.2)), although, in fact, the model under consideration is a "probabilistic" modification of the model 4.1. The constancy of the coefficients leads to the exponentiality of the solution, and the learning rate $\gamma = \alpha + \beta$, as before, is determined by the magnitude of the coefficients α and β .

A significant number of works, especially foreign authors, are devoted to statistical models of learning. In most of them, IN is understood precisely as "... a systematic change in the probability of a reaction" [22, p. 395]. Here is one of the sets of requirements for statistical models:

1. "The dynamics of the averaged learning indicator is described by a curve having a negative acceleration in its final phase and tending to some

constant asymptote" (note that in this section, slow asymptotic behavior is required only in the final phase, that is, for example, the presence of an initial plateau is allowed - D.N.).

2. "A smooth average curve is the result of averaging ... and the asymptote of the observed LC represents only the point of statistical equilibrium" [22, p. 397].

It should be noted that the obtained solution of equation (4.2) is consistent with the results of experiments with many animals (in most cases, with rats) [16], people [28, etc.] and stochastic automata [73, 74].

The exponential form of the LC is due to the linearity of dependences (4.1) and (4.2) and the constancy (stationarity) of the coefficients α and β . In the following model, this dependence is already taken nonlinear. •

Model 4.3. [16].

D. The change in the mismatch (for example, the dependence of the probability of the correct reaction on the number of repetitions) of the system over time.

H. At each step, the change in the mismatch is proportional to the current value of the mismatch and the difference between some final mismatch α and the current. The mismatch dynamics satisfies the Bernoulli differential equation

$$(4.3) \quad \frac{dx(t)}{dt} = \beta x(t) (\alpha - x(t)),$$

where α and β are some constants.

T(C). At the starting point x , the solution is the logistic curve:

$$x(t) = \alpha x^0 / (x^0 + (\alpha - x^0) e^{-\alpha \beta t}).$$

A. The presence of a "braking add-on" in (4.3) in comparison with (4.1) and (4.2) leads to the fact that the LC is not exponential, but logistic — an inflection point appears. The learning rate, in contrast to previous models, depends not only on the proportionality coefficient between the rate of change of the mismatch and the current value of the mismatch, but also on the magnitude of the final mismatch. •

Model 4.4. [26, 32, 33].

D. The classic axiomatic model of iterative learning is the well-known system of postulates of C. Hull for the behaviorist model S-R-S (the basis of training is the strengthening of stimulus-reaction relationships).

H(A, C). The law of skill formation (IV postulate) states that if the reinforcements are uniform (the uniformity of samples is an important characteristic of iterative learning) follow one after another, and everything else (external conditions and learning objectives) does not change, then as a result the strength of the skill $x(n)$ will increase with increasing number of tests according to the equality:

$$x_n = 1 - 10^{-\gamma n}.$$

A. Note that the curve of forgetting according to the VIII postulate is also an exponential curve [33]. •

Model 4.5. [4, 5].

D. A "generalized training model" (for example, training a human operator). The variable x is the probability that the trained system has formed an adequate model of the external environment.

H. From an analog of the principle of least action (see also the models in Section) it follows that the change in probability satisfies the differential equation [5]:

$$(4.4) \quad \frac{dx(t)}{dt} + \alpha x(t) = \beta.$$

Note that sometimes equations of type (4.4) are called the "law of reinforcement of the statistical theory of learning." In [21] this law is written as

$$x_n = x_{n-1} + \alpha (1 - x_{n-1}),$$

which corresponds to $\beta = \alpha$ (or (4.2) with $\beta = 0$; if $x^0 = 0$, then $x^\infty = 1$ [4]).

T(C, A) – see model 4.2. •

Many researchers initially postulate a slow-asymptotic type of SC and use it in the future for quantitative analysis, development of various recommendations, etc. [43 et al.].

In almost all models of this section, it is assumed that the mismatch of the system satisfies the linear differential equation with constant coefficients. Moreover, the linearity and stationarity of the coefficients are sufficient (but not necessary) conditions for the exponentiality of the solution.

5. Models - analogies of physical phenomena and technical systems

The iterative learning models discussed in this section, proposed by various authors, are based on analogies of physical phenomena and the principles of functioning of technical systems. Many of the analogies used are rather arbitrary and the adequacy of assumptions to the valid patterns that occur in biosystems can cause justifiable objections.

Model 5.1. [18].

D. In some models of the nervous system, the brain is considered as a technical system for pattern recognition, the parameters of which depend on the electrical characteristics of nerve fibers.

H. The neuron process is a long RC-chain (RC-line consisting of a capacitor and a resistor).

T. If U_{in} is the voltage at the input of the RC-circuit, $U_{out}(t)$ is the voltage at the output, then the connection between them, by virtue of Kirchhoff's laws, is described by the differential equation:

$$C \frac{dU_{out}(t)}{dt} = \frac{U_{in} - U_{out}(t)}{R},$$

where C is the capacitance of the capacitor, and R is the resistance value.

C. The output voltage varies exponentially. Since the temporal characteristics of the processes of signal transmission and propagation in the nervous system are determined by exponential transfer functions with a characteristic time $\tau = RC$, so $\gamma = 1/\tau$ will determine the rate of transitional (adaptive) processes in the system, that is, be described by an exponential dependence.

A. The difference in the amplitude of the signal (stimulus) in the considered model is described by a law that practically coincides with the Weber-Fechner law. The output voltage of the circuit - the main characteristic of the model - satisfies the linear differential equation (see the fourth section). •

Model 5.2.

D(H). By analogy with the mechanisms of radioactive decay in physics, suppose that the mismatch of the learning system is determined by the mismatch of elements, each of which may have either some initial mismatch or some final mismatch. The system mismatch is a function of the number of elements having nonzero mismatch, and the decrease in mis-

match occurring for each element in steps is a probabilistic process characterized by a constant (independent of time and number of elements) probability γ "zeroing" of the element mismatch per unit time.

F. The number of elements $N(t)$ with nonzero mismatch at time t satisfies the equation $N(t + \Delta t) = N(t) - \gamma N(t) \Delta t$.

Passing to the limit in Δt , we obtain the differential equation

$$(5.1) \quad \frac{dN(t)}{dt} = -\gamma N(t).$$

C. Solution of equation (5.1) is

$$(5.2) \quad N(t) = N_0 e^{-\gamma t},$$

where N_0 – the number of elements in the system (at time zero, all elements had a maximum (initial) mismatch).

A. The constant γ , characterizing the half-life, characterizes the speed of learning. The greater the probability of reducing the mismatch of an element per unit time, the higher the learning rate.

Note that the assumption of the sameness for all elements and the stationarity of the probability of "decay" is significant.

It is also important that the above equation for $N(t)$ is satisfied not only by the mechanisms of radioactive decay, but also by the processes of bacterial growth, pharmacokinetic processes, most kinetic schemes of chemical reactions (including the law of acting masses), etc. from time to time, the macroscopic characteristics in all these cases turn out to be exponential simply because the behavior of any element is probabilistic, and the statistical characteristics of the processes (decay, growth, etc.) are independent of time and the history of the system. This statement of stationarity, which underlies the description and explanation of the mentioned class of processes, is an assumption consistent with experimental data. •

Model 5.3.

D. Each element of the learning system has its own regulator, striving to reduce its mismatch. The mismatch of the system as a whole is a monotonic function of the mismatch of elements.

H. Each regulator is characterized by a constant relative error in (to require the constancy of absolute error seems illogical, since the regulator must be universal [7]). At the n -th step, the controller randomly transfers the element from the state x_{n-1} to the state x_n uniformly distributed in the $\delta = \delta(x_{n-1})$ -neighborhood of zero mismatch.

T(C, A). With a sufficiently large n , the learning curve - the average mismatch of the elements - is a decreasing exponential function. The type of LC is due to the constancy of the relative error of the regulator and the assumption of probability distributions (compare with the change in information when measuring values with an error [14]). •

Model 5.4.

D(H, T, C). The training system is a set of first-order controllers (that is, first-order aperiodic units that control the magnitude of the variable and its rate of change), similar to those used in automatic control. Transfer function (response to pulsed input) of each element is

$$h(t) = 1 - \exp(-\gamma_i t).$$

A. It is interesting to note that the second-order aperiodic link (which controls the value of the variable and its first two derivatives), which can be considered as a serial connection of two first-order aperiodic links, has a logistic transfer function. Within the framework of this model, logistic learning curves can be considered as the LC of a hierarchical system consisting of two subsystems, the results of iterative learning of each of which is described by an exponential curve. •

Model 5.5. [4].

B. The probabilities of finding the system in certain states are investigated. Let the learning system have two possible structural states s_1 and s_2 . We denote the probabilities of finding the system in these states

$$p = \text{Prob} \{s_1\} \text{ and } q = \text{Prob} \{s_2\}; q = 1 - p; p' = \frac{dp(t)}{dt}.$$

H. By analogy with mechanical systems, we assume that the system is described by two functions of time, one of which we will arbitrarily call the level of organization ("potential") of the system:

$$(5.3) V(t) = \alpha p^2(t),$$

and the second - the "kinetic energy" of the system:

$$(5.4) T(t) = \int (p')^2 d\tau.$$

We note that $V(t)$ and $T(t)$ correspond to the potential and kinetic energy of a mechanical system whose phase variable is $p(t)$. The function $K = T - V$ is the "total energy of the system." Next, we introduce the following assumption: "In order for the dynamic process of changing the level of organization of the system, due to internal causes or actions of the

environment, to be optimal, it should apparently obey a principle similar to the principle of least action" [4].

T. Substituting (5.3) and (5.4) into the Lagrange equation and solving it, we obtain

$$(5.5) p(t) = 1 - e^{-\gamma t},$$

где

$$(5.6) \gamma = \alpha / \beta.$$

C. "The optimality of living systems lies in the exponential laws of probability change ..." [4].

A. It should be recognized that today the model described above is one of the most elegant and beautiful (if these terms can refer to mathematical models).

Without diminishing the merits of the model and its significance, we will try to restore the course of reasoning of its author.

First, it is known from experiments that the probabilities in the process of IL change in most cases according to an exponential law. Secondly, there must be general laws for the functioning of living systems. Since the principle of least action has sufficient generality (at least for mechanical systems), we transfer it to living systems.

And then everything is quite simple - we write the corresponding equations and investigate what the structure of the "potential" and "kinetic energy" should be so that the solution satisfies (5.5). It turns out that the only construction leading to the desired result is (5.3) and (5.4). It should be noted, however, that the choice of the initial conditions and (5.3)-(5.4) is not trivial. Moreover, meaningful interpretations of (5.6) as learning speeds are also difficult.

This model demonstrates very well the simultaneous application of both the direct method of constructing IL models (when assumptions are introduced and conclusions are drawn from them that coincide with the experimental data) and the converse (in which those assumptions and hypotheses about the functioning mechanisms of the system under study that lead to the desired result are sought) •

Thus, the above iterative learning models, constructed by analogy with the principles and laws of the functioning of physical and technical systems, use the "generalization" of a number of physical laws. As a rule, an assumption is made that the laws (in most cases, conservation laws) formulated for a certain class of systems of animate and inanimate nature (and

valid for describing learning systems at a certain microlevel of consideration) remain valid for a “macroscopic” description of these systems. The validity of this assumption in most cases, unfortunately, is not yet supported by experimental confirmation.

6. Information theoretical models

A significant part of the iterative learning models described in the literature is made up of models based on the consideration of information processing in trained systems. What unites these information-theoretical models is that, in almost all of them, it is assumed that the capabilities of the learning system for the transmission and processing of information (the amount of information transmitted, processed, acquired, etc. per unit time) are limited [13, 35, 40, 46 and other]. For example:

"... the average time required to clearly understand the meaning of a certain signal and the correct response to it increases in proportion to the average information contained in this signal. Based on this, we can assume that in the case of fairly regular events that are characterized by a certain statistical stability, the message about the occurrence of such an event is transmitted through the senses and the central nervous system on average over a time proportional to the information contained in this message. ... transmission community In a living organism, it happens so that, over the same time, the same amount of information is transmitted on average "[80, p. 115].

A special case of the assumption of the limited possibilities of a person in the processing of information is the well-known Hick law, which establishes proportionality (in a certain range) between the amount of processed information and the signal uncertainty; when the last certain threshold value is exceeded, the amount of processed information remains constant.

There are two types of information - related (initial, a priori information embedded in the structure of the system) and free. The learning process can be interpreted as follows: "... free information gradually becomes connected, there is a process of" learning " - increasing the initial organization of the system, increasing the amount of related information" [81, p. 15]. Training can also be understood as "... the development of a system without increasing the elemental composition, increasing the value of information by establishing additional links" [19, p. 193], and modification of the structure of goals in most cases causes only quantitative rather than qualitative changes [37].

The information received at the input of the system or its subsystem can be used, in particular, as follows:

- 1) direct reaction;

2) remembering previous situations in order to select the most successful reactions of a direct type;

3) remembering external influences in order to extrapolate them and identify a rational response to extrapolated external influences;

And, finally, the most common fourth case is the creation of models of the outside world and obtaining a forecast based on the functioning of models [41].

Almost all the iterative learning models discussed in this section are based on the above points.

Model 6.1. [4].

O. In [24], an approach was proposed to define the concept of the organization of a system and its complexity [17] through entropy. The correspondence between the complexity and organization of the system and the complexity and organization of the environment is established by the principle of adequacy.

Various formulations of the principle of adequacy are known [4]. For example, the capabilities (complexity, throughput, etc.) of the control system determine the limits of "controllability" of the control object, no matter how great its own capabilities are (the inverse relation is extremely rare in biology). In other words, "in order for the system to function successfully in the environment, its complexity and organization must be adequate to the complexity and organization of the environment" [4].

In [4], the principle of dynamic adequacy was proposed: "... with a change in the complexity and organization of the environment, the biosystem constantly strives to achieve a new level of adequacy in complexity and organization with the environment, minimizing the time, cost of matter and energy."

H. In particular, in [4] the following assumption is introduced (which in one form or another is used in almost all theoretical information models of the IL: change in entropy in the learning system - (the amount of information processed received, transmitted etc. by the system) in proportion to the change in the entropy of the environment.

T(C, A). The proportionality coefficient depends on the capabilities of the system - the bandwidth of the information transfer channels, the maximum allowable rate of change of the parameters of the elements, etc., and if the proportionality coefficient and the amount of information received per unit time are constant (do not depend on time), then the dynamics of

the system, obviously, is described by an exponent (see below in more detail). If training is considered as a process of obtaining information, then in the learning system there is a phased elimination of uncertainty due to information coming from the external environment [44]. •

Model 6.2. [62].

D. The process of processing information by a trained system.

H. Suppose that information flows satisfy the equation

$$(6.1) \quad \frac{dI}{dt} = \alpha \frac{dJ}{dt} + \beta J,$$

where I is the amount of incoming information, J is the amount of absorbed information, α and β are constants that characterize the learning system and determine the speed of learning.

Equation (6.1) indicates that the rate of assimilation of information is proportional to the rate of receipt of information and decreases (also proportionally) with the growth of information already received.

Suppose that the amount of information entering a unit of time is constant:

$$(6.2) \quad I(t) = \theta t.$$

T(C). The solution of (6.1) within the framework of the assumption made has the form

$$(6.3) \quad J(t) = \delta(I - e^{-\gamma t})$$

where

$$(6.4) \quad \delta = \theta / \beta, \quad \gamma = \beta / \alpha.$$

A. Assumptions about the constancy (or limitation) of the amount of information received or processed by the learning system per unit of time are used in almost all information-theoretic models of iterative learning, and in most of them they have exactly the form (6.2). In this model, to obtain expression (6.3), it was necessary to introduce a rather specific hypothesis about the relationship between incoming and assimilated information. It is interesting to note that the learning rate determined by the constants α and β does not depend on the rate of receipt of information θ - an external parameter, but is determined only by the parameters of the system itself. •

Model 6.3. [57].

D. Memorization and storage of information in human memory.

H. Information flows obey the relation

$$(6.5) \quad \frac{dJ}{dt} = \frac{dI}{dt} - (J - J^\infty) / T,$$

where J is the amount of assimilated information, $\frac{dJ}{dt}$ is the rate of assimilation of information, $\frac{dI}{dt}$ is the rate of supply of information, T is the time constant (the characteristic time that determines the speed of learning) of the information processing process by the human memory, J^∞ is the limit value of the acquired information (compare with (6.1))

T(C, A). Assuming $\frac{dI}{dt} = \theta = \text{Const}$ (constancy of external conditions), solution (6.5) has the form

$$(6.6) \quad I(t) = \delta(I - e^{-\theta t}),$$

where $\delta = I^\infty + \theta T$, $\gamma = I / T$ (compare with (6.3)). •

Model 6.4. [65].

D. The process of accumulating information and forgetting it.

H. With a constant amount of information coming in per unit of time, the “ideal memory” remembers all the information. In real memory, the amount of information stored per unit time decreases with the growth of already stored information (delayed asymptotic behavior). After the end of the learning process, the ideal memory stores information indefinitely, and in real memory the amount of information decreases monotonously (forgetting) after the end of the learning process, and the current forgetting speed is proportional to the amount of information $I(t)$ available at the moment (slow asymptotic behavior, see Fig. 6.1).

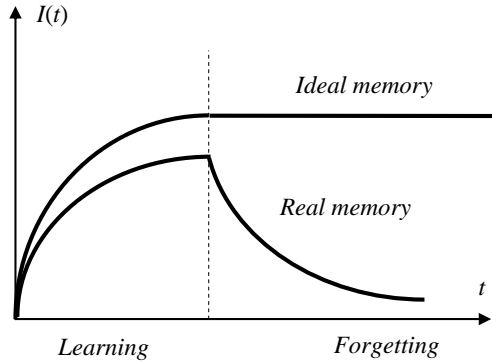


Fig. 6.1. The amount of remembered information

T(C, A). If the “memory equation” is represented by a linear integral equation, then the qualitative conclusion will be the same as when using equations (6.1) and (6.5) in models 6.2 and 6.3 [65]. •

Model 6.5. [71, 72].

D. Processing information by a human operator.

H(T, C, A). The exponential dependence of the quality of the operator's work on the training time is postulated. •

Model 6.6. [64].

D. Information processing by the operator (in the human-machine system) during training and in the process of professional activity.

H. The amount of information I processed by the operator in the process of his activity corresponds to a change in his entropy: $I = \Delta H$. Therefore, the disorder in the activity of the operator W (the number of possible states of the system under study, the logarithm of which determines the entropy) depends on time as follows:

$$(6.7) \quad W(t) = W_0 e^{-\beta t}.$$

Assume that $I(t) = \alpha t$, where t is operator training time, α is a constant characterizing the training system. Define the quality of the operator as follows

$$Q(t) = Q_{max} (1 - W(t)).$$

T(C). Then

$$(6.8) Q(t) = Q_{max} (1 - W_0 e^{-\gamma t}),$$

where $\gamma = \alpha \beta$.

A. The exponential nature of the LC is due to the choice of entropy and information as the characteristics of disorder, specific (in particular, linear) dependences of the characteristics of the operator's activity on disorder and the assumption of a linear increase in the amount of accumulated information. In the model under consideration, the speed of learning depends both on the rate of information flow in the learning process, and on the characteristic time of disorder change.

It should be noted that in [64] there were three stages of training:

1. The initial "running-in" of the human operator to this mode of operation.
2. "Development" of effective characteristics within the framework of a fixed regime (actually the stage of iterative learning).
3. Activities characterized by statistically stable characteristics.

In this case, the dependence of the error on time can be represented schematically by the curve shown in Fig. 6.2. •

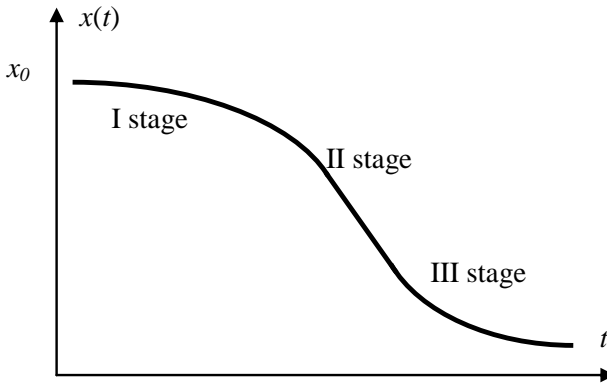


Fig. 6.2. Dependence of operator error on time

Model 6.7. [27].

D. Information processing during the training of the perceptron (pattern recognition system, which can be considered as a model of memorization and learning in living systems).

H. For the correct recognition of the i -th image, it is necessary and sufficient that it be shown at least once to the perceptron in the learning process.

T. For n random (equally probable) displays of images, the probability of occurrence of one of the N samples is

$$(1 - 1/N)^n \cong \exp(-n/N).$$

C. Then the full effectiveness of training (the probability of correct recognition depending on the duration of the learning phase)

$$p_n = 1 - e^{-\gamma n}.$$

where $\gamma = 1/N$.

A. Compare with the model 5.2. In this model, as in 5.2, the probability of decreasing the mismatch of elements (each element is “responsible” for remembering one image) is characterized by a constant probability γ of “zeroing” its mismatch per unit time (the probability that the corresponding image was shown and remembered). The learning system is supposed to be quite passive, so the learning speed is inversely proportional to the number of possible options N . •

Model 6.8.

D. The system being taught has a communication channel through which information is received from the external environment during the learning process, and the more information is received by the system, the less is the mismatch.

H. There is interference on the communication channel, whose bandwidth is limited [63]. At each step, all information that is not yet received by the system is sent, and each time the system receives only some fixed part of it undistorted.

T. Suppose that for successful learning, the system must receive complete information I . At the first step, all information is sent, undistorted “reaches” γI ($\gamma < 1$).

At the second step, information is sent in the amount of $(1 - \gamma) I$, from which the system receives $\gamma(1 - \gamma) I$, etc. The amount of information received by the system in $n > 2$ steps is determined by the expression

$$(6.8) J_n = (1 + (1 - \gamma) + (1 - \gamma)^2 + \dots + (1 - \gamma)^{n-1}) \gamma I,$$

i.e. $J_n = \gamma I (1 - (1 - \gamma)^n)$.

Other interpretations are possible. Let, for example, all information I be sent at each step. Then the amount of received information changes over time as follows:

$$(6.9) J(t+\Delta t) = J(t) + \gamma(I - J(t)) \Delta t.$$

C. Solution of (6.9) is

$$(6.10) J(t) = I(1 - e^{-\gamma t}).$$

A. This type of solution is due to the proportionality of the amount of new information received by the system to the amount of information that remains to be transmitted. In other words, this property (assumption) can be interpreted as follows: the ability of the system to absorb (remember) information decreases in proportion to the amount of stored and processed information.

In this case, it is critical (in order for the solution to have the form coinciding with (6.10)) that the proportion between the received part of the information and the already accumulated one remains constant in time. It should be noted that within the framework of this model, simply assuming limited bandwidth of the communication channel would lead to completely different conclusions (the amount of accumulated information would grow linearly, etc.). The learning rate in the model under consideration is determined by the channel capacity γ - the more information comes without distortion, the higher the learning speed. •

Model 6.9.

D. Consider a complex learning system in the form of a set of elements (their number is denoted by N), the combined actions of which lead to the achievement of some fixed goal.

Suppose that each element is characterized by a finite set of its admissible states $S_i(t)$ (the number of elements of the set S_i is $n_i(t)$), in one of which it can be at time t , $i = \overline{1, n}$. The number of independent states of the system as a whole (described by enumerating the states of its non-interacting elements) is equal to the product of the number of admissible states of all elements.

H. Suppose that learning consists in reducing the number of permissible states of each element to a certain minimum, that is, in leaving one or more fixed states corresponding to the problem being solved. The purpose of training for the system is to minimize the number of its permissible states. A decrease in the number of permissible states of each element occurs as it receives information.

Entropy of the i -th element (its disorder) is

$$(6.11) H_i(t) = \ln n_i(t).$$

The amount of control information $\xi_i(t)$, received by the i -th element at time t , reduces the uncertainty:

$$(6.12) \quad \frac{dH_i(t)}{dt} = -\xi_i(t), \quad t > 0.$$

Suppose that there is an absolute limit to the amount of regulatory information arriving at each moment: $\xi_i(t) \leq \gamma_i, \forall t \geq 0$. In the general case, at time t , $\xi_i(t)$ belongs to the interval $[0; \gamma_i]$ ($\xi_i(t) \equiv 0$ corresponds to the fact that the i -th element at the moment t is not being trained).

T. We study how the number of permissible states of elements will change over time. Substituting (6.11) into (6.12) and solving the corresponding differential equation, we obtain

$$(6.13) \quad n_i(t) = n_i^0 \exp\left(-\int_0^t \xi_i(\tau) d\tau\right), \quad i = \overline{1, n}, \quad t > 0,$$

where n_i^0 is the number of permissible states of the i -th element before the start of learning. The integral in the exponent corresponds to the accumu-

lated information: $I_i(t) = \int_0^t \xi_i(\tau) d\tau$.

C. Let us consider how the number of admissible states of the system as a whole will behave in time, reflecting, by virtue of the assumption introduced above, the effectiveness of learning:

$$(6.14) \quad n(t) = \prod_{i=1}^n n_i(t) = n^0 \exp(-I(t)),$$

where $n^0 = \prod_{i=1}^n n_i^0$,

$$(6.15) \quad I(t) = \sum_{i=1}^n I_i(t).$$

If we assume that the characteristics of the elements and the rate of receipt of information are constant, that is, the amount of information processed by each element per unit of time is constant: $I_i(t) = \theta_i t$, then (6.14)

passes into the classical exponent with the exponent $I(t) = t \sum_{i=1}^n \theta_i$.

A. The hypothesis of a monotonic decrease in the number of admissible states does not reduce the generality of the above reasoning, since if they grow, an expression of the form

$$n(t) = n^{\infty} (1 - e^{-I(t)}),$$

with about the same intermediate calculations.

The results of models 6.2, 6.3, 6.5, 6.6, and 6.8 can be considered as special cases of model 6.9.

In all the models of this section, the learning rate is determined by the amount of information accumulated, therefore, to increase the learning speed, within the framework of the model under consideration, it is advisable to choose the highest possible rate of information transfer. However, it should be borne in mind that in real systems, exceeding a certain threshold (for the trained system), volume of incoming information can have a negative effect and reduce the effectiveness of learning (analogue of the effect of interference of skills). •

Thus, in the theoretical information models of iterative learning, the exponential nature of learning curves is due to the constancy of the amount of information processed, transmitted, assimilated, etc. system elements per unit time.

7. Models - analogies of cybernetic systems

The difference between the iterative learning models discussed in this section and those described above is that the objects of research are not living systems, the study of which is based on hypothetical analogies and assumptions about the relationship between the parameters of the elements and the learning system, and cybernetic systems are automata, algorithms, neural networks, etc. In other words, when constructing mathematical models of iterative learning of biological systems, we used above analogies with physical phenomena, these or those intuitive assumptions, etc. In models - analogies of cybernetic (abstract-logical models that are not materially implemented, unlike technical) systems, the principles of functioning of the latter, on the one hand, are transferred (at the level of hypotheses) to simulated systems, and on the other hand, many cybernetic systems use analogies with living systems .

The separation is not accidental. For example, finite state machines and neural networks are widely used in control theory, applied mathematics, and other fields of science, not only as models of living systems, but also as objects that deserve independent study and are used in the synthesis of control systems, pattern recognition, etc. [56, 61]. To the same class of models we include models that use analogies with optimization methods - there are a number of IL models in which it is assumed that nature "uses" one or another algorithm to reduce, for example, the mismatch value. On the other hand, if we want to draw some conclusions about the behavior of humans and animals in iterative learning based on an analysis of the behavior of, for example, a neural network, then we need to understand what relation the studied cybernetic system has to the network of neurons in the human brain.

At the same time, however, one must clearly understand that artificial systems behave in one way or another not by themselves, but in strict accordance with the rules and algorithms that were laid down in them by the person who created the system.

The first use of extremum search methods in the analysis and modeling of the behavior of biological systems is, apparently, the ravine method, in which all variables (system parameters) are divided into two qualitatively different classes - significant and non-essential. Some of them are characterized by the fact that when they change, the value of the minimized function changes quite quickly (descent along the slope of the "ravine" -

the surface of the function), and others by a rather slow change of the minimized function (descent along the inclined bottom of the ravine). Accordingly, to achieve the minimum as quickly as possible, it is necessary to move along the bottom of the ravine as quickly as possible (note that here and in the course of the subsequent presentation we will not discuss the locality of the algorithms, their convergence, etc., confining ourselves to a qualitative analysis).

Model 7.1.

D(H, T, C). Suppose that the algorithm for minimizing the mismatch uses the method of finding the root (of some function $f(x)$ on the interval $[a; b]$) by dividing the segment in half. The upper bound for the mismatch (depending on the number of iterations) is given by the expression $x_n \leq (b - a) / 2^n$, i.e., $x_n \leq \alpha e^{-\gamma n}$, where

$$\alpha = \exp(\log_2(b - a) \ln 2), \gamma = \ln 2.$$

A. Approximately exponential convergence (for sufficiently "good" functions) have not only dichotomous root search methods, but also many others. •

Model 7.2.

D(H). Suppose that a system mismatch at time n is defined as the arithmetic mean of the current mismatch values of all N elements.

Let the mismatches of all elements at the initial moment of time be equal to unity, non-negative at any moment of time, and at the n th moment of time the mismatch of the i -th element $x_i(n)$ can take any value less than $x_i(n - 1)$ with equal probability.

T(C). Then, if you define the mismatch of the entire system as $X_N(n) = \frac{1}{N} \sum_{i=1}^N x_i(n)$, then, if the number of elements is large enough, then the system mismatch is $X_n = X_{n-1} / 2$, $n = 1, 2, \dots, X_0 = 1$.

A. The assumption of non-growth of the mismatches of the elements is fully consistent with the well-known principle of "not missing what was achieved" [7]. At the same time, the use of the arithmetic mean as the value of the mismatch of the system and the assumption that the permissible values of the mismatch of the elements are equally probable do not seem very justified. It is worth noting some proximity of the model under consideration to models 5.1 and 8.4. •

Model 7.3. [9].

D. A technical system whose variable characteristics are probabilities (of certain actions, states, reactions, etc.).

H. Depending on the “success” or “failure” in step n , in step $n + 1$, the probability p is defined as follows:

$$p_{n+1} = \begin{cases} p_n + \alpha (1 - p_n) \\ p_n - \beta p_n \end{cases}.$$

T(C). Suppose that if the correct action is chosen at the n -th step (with probability p_n), then the probability of “success” is p (respectively, “failure” - $(1 - p)$). If the wrong action is chosen (with probability $(1 - p_n)$), then the probability of “success” is q . Then the expectation of “success” at the $(n + 1)$ -th step is equal to $V_{n+1} = V_n (p_{n+1} p + (1 - p_{n+1}) q)$.

Substituting the law of probability variation, we find that V_n changes exponentially with time (see model 4.2).

A. The exponential form of a curve reflecting a change in expected “success” is due to a linear change in probability. In the 50-60s, during the rapid development of cybernetics, a significant number of the most diverse learning machines were built: conditional probability machines [9], learning matrices [66], C. Shannon’s “mouse” (labyrinth model), “turtle” of G. Zemanek, the “speculative machine” (analogue of the unconditioned reflex) and “CORA” (analogue of the conditioned reflex) G. Walter [76] and others.

Most of them used linear laws of change of variables (in contrast, for example, to non-linear laws used in the homeostat by W.R. Ashby [7]). Moreover, when studying the general laws of adaptation and learning processes in automatic systems, many laws of instruction (for example, linear algorithms of optimal learning) were also chosen linear. •

The exponential form of a curve reflecting a change in expected “success” is due to a linear change in probability. In the 50-60s, during the rapid development of cybernetics, a significant number of the most diverse learning machines were built: conditional probability machines [9], learning matrices [66], K. Shannon’s “mouse” (labyrinth model), “turtle” of G. Zemanek, the “speculative machine” (analogue of the unconditioned reflex) and “CORA” (analogue of the conditioned reflex) G. Walter [76] and others.

A large class of learning automata are the so-called finite probabilistic/stochastic automata with variable structure. A finite state machine is understood to mean an object having some internal states, the input of which may receive external influences and the output parameter of which can take one of a finite number of values [38, 74]. The internal states of the automaton change with a change in the input parameters, and the output states with a change in the internal states. For our analysis, the ability of the automaton to "independently" change its structure is important - the transformation "input" - "internal state", "input, internal state" - "output" (of course, the machine does not change these laws at its discretion, but in accordance with algorithm into it), functioning in a non-stationary environment. This ability allows one to talk about adaptability of behavior, effects of collective behavior (games of automata, hierarchical learning automata [39, 73]) and the presence of some kind of learning (understood in this case as the accumulation and processing of information about the external environment and the development of appropriate laws of behavior in these specific conditions [73]).

Model 7.4. [39, 73].

D. A probabilistic automata at time t performs the i -th action (selects the i -th output state) with probability $p_i(t)$, $i = \overline{1, k}$, where k is a finite number of output states. The purpose of the automata is to maximize the gain, depending on its actions and the state of the environment. The "variability" of its structure means the possibility of changing probabilities. It is clear that if under the given conditions (under the given state of the environment) the "correct" action was chosen that led to a positive gain, then the probability of choosing this action should be increased, and the probabilities of choosing other actions should be reduced accordingly, since the normalization condition must be fulfilled (compare with the "labyrinth" model 4.2).

H. Suppose that the probabilities of the choice of actions i and j vary according to such a law $\Delta_{\pm} p_i(t)$, that

$$\begin{aligned} p_i(t+1) &= p_i(t) \pm \Delta_{\pm} p_i(t), \\ p_j(t+1) &= p_j(t) \pm \Delta_{\pm} p_j(t), \quad j \neq i, \\ \Delta_{\pm} p_i(t) + \sum_{j \neq i} \Delta_{\pm} p_j(t) &= 0. \end{aligned}$$

T(C, A). If the law of change $\Delta_{\pm}p_i(t)$ is linear in $p_i(t)$, we obtain an exponential sequence. In the general case, of course, no purely exponential curve will be observed, however, in most cases, during simulation, approximately exponential slow-asymptotic curves were observed, for example, the average gain on the number of games played [38, 39]. •

Another vast class of cybernetic systems that claim to simulate the phenomena and processes that occur in biological systems are the so-called *neural networks*.

Algorithms for learning neural networks can conditionally be divided into deterministic algorithms and random search algorithms. In fact, training a neural network is nothing more than the task of minimizing the multi-extreme function of many variables [31]. The number of various training methods known today (minimization algorithms) and various network designs (their architectures) is at least several dozen. We will look at some common approaches to learning neural networks without going into details.

A neural network consists of several layers of neurons that have logistic or some other sigmoid-like transfer functions. The outputs of the neurons of each layer are fed to the inputs of the neurons of other layers with specific weights. The weight of the “connection” is the number $w(i, j)$ by which the output signal of the i -th neuron is multiplied before summing at the input of the j -th neuron. Learning a neural network consists in selecting (sequentially changing) the weights of neurons corresponding to the problem being solved (signal recognition, minimization of function, etc.). The training takes place as follows: certain signals are supplied to the neural network, the output signals of the network are compared with standard values, and based on this comparison, weights are adjusted.

Quite common algorithms for changing weights are the backpropagation algorithm - first, the weights of the neurons of the last (output) layer, then the penultimate one, etc. change, and the so-called random multistart (more precisely, its modifications - the starting point is selected, the next point is determined by adding to the initial, for example, a Gaussian random vector and an “inertial additive,” the values of the error function at these points are compared, etc.).

In fairness, it should be noted that in the general case, the weights of individual neurons and their errors do not always change in a slow-asymptotic manner. However, the general error, which is most often calculated as the average error of neurons, in most cases varies approximately

exponentially (in particular, when using the gradient descent method). It is clear that the dynamics of error depends both on the method of learning used and on the specifics of the minimized function. The rate of convergence to the minimum point of the error function (the rate of learning a neural network) depends on the algorithm for changing the weights of neurons, which, in turn, is laid down by the designer.

Thus, when teaching cybernetic systems, the exponential nature of the corresponding LC is due to the linear law of variation of the internal parameters of the system and / or a large number of its constituent elements.

8. Models of collective behavior

This section discusses iterative learning models based either on the results of experimental observations of the interaction of team members, or on analogies with the principles used in formal models of collective behavior.

Model 8.1. [7].

One of the first models of the adaptive interaction of elements is the Ashby homeostat, which serves as a good illustration of the possibilities of using ultrastable dynamic systems in modeling the properties of the nervous system. It should be recognized that since the study of the homeostat focuses on the adaptability of behavior, its “learning curves” in some cases are not slow-asymptotic. This model is so famous and studied in detail that we restrict ourselves to a reference to the source [7]. •

Model 8.2. [49].

D. The homeostat model can be used to analyze the group activities of operators. In fact, the difference from the previous model is that the compensation of influences (external to a particular operator) is carried out not due to physical feedback (device device), but due to the purposeful activity of each operator, taking into account the actions of the others.

H(T,C). The matrix equation of the “Homeostat” has the form: $\chi = A U$, where U is the matrix of positions of the control knobs, χ is the matrix of positions of the hands of the instruments, A is the matrix characterizing the structure of the “homeostat” and the values of the coefficients of mutual coupling (the readings of each device are a linear combination of the positions of the control knobs). Depending on the method of connecting the operators (a ring, star, chain, etc. were used) and the number of operators, the difficulty of the tasks being solved is determined.

A. For various structures, the difficulty of the problem to be solved depends significantly on the number of operators. The assumption of a linear relationship significantly simplifies the model. In this case, again, due to adaptability, the dynamics of the system is not always described by a delayed asymptotic curve. •

Model 8.3. [58].

D. Self-organization parameters in a group of three subjects.

H. Denote by H_{\max} - the maximum value of the entropy of the system, $H(t) \leq H_{\max}$ - current entropy value, $h = H_{\max} - H$ - the amount of accumulated information. Suppose that the rate of accumulation of information (increment of information per iteration or per error) is constant (see Section 6) and that the residual entropy is evenly distributed between the identifiable objects.

T(C). In accordance with the accepted assumptions, if $x(t)$ is the total number of errors in time t , then $\frac{dx}{dt}$ is the probability of error at the time t ,

$$\frac{dH}{dt} = \gamma, H(t) = -M \ln(1 - \frac{dx}{dt}).$$

If $x(0) = 0$, then $H = \gamma x$. As a result, we

obtain the following equation of the theoretical total error curve:

$$x = H_{\max} / \gamma - M / \gamma [\exp(H_{\max} / M - 1) \exp(-\gamma t / M) + 1].$$

A. The validity of a number of assumptions accepted by the author of this model is not obvious, some statements (especially formal ones) need explanation. Nevertheless, [58] is considered one of the classic works on experimental and formal research of self-organization processes in collectives. Note that the resulting expression determines the dependence of the accumulated error on time. The curve of the current value of the mismatch will be logistic. •

From our point of view, game-theoretic models of iterative learning, or rather, models using the results of the theory of collective behavior, have sufficient generality.

Before considering specific models, we will describe the general principles. Let the system consist of n elements, each of which can be in the state $s_i(t) \in \Omega = [s_i^-, s_i^+]$. Suppose that the state of the entire system is uniquely described by the state vector of elements:

$$s(t) = (s_1(t), s_2(t), \dots, s_n(t)), s(t) \in \Omega = \prod_{i=1}^n \Omega_i, \forall t \geq 0.$$

The quantity $h(\tau) = \{s(t) \in \Omega \mid \tau < t\}$, that is, information on the strategies of all the elements selected up to the moment τ will be called the history of the game.

Consider how the elements will behave. Suppose that there exist some functions $\varphi(\cdot) = \{\varphi_i(s)\}$, which we will call the objective functions of the elements, reflecting the interests of the elements (each element seeks to

maximize the value of its objective function). Note that the objective function of each element in the general case depends not only on its own state (the strategy chosen by it or assigned to it by the “control device”), but also on the states of other elements, that is, there is a play of elements (for example, each element can tend to minimize the indicator function [51]). We will assume that this game is non-cooperative, that is, each element chooses a strategy on its own, without being able to agree with the other elements.

By successively changing their strategies, the elements strive to reach a certain equilibrium point. In game theory, there are several concepts of equilibrium. If we consider the game of elements to be non-cooperative, then it is advisable to consider the Nash equilibrium (as such a combination of strategies, a single deviation from which is not beneficial to any of the elements). For our analysis, the primary is not the concept of equilibrium, but the principles of the behavior of the elements. By the principle of behavior of the i -th agent we mean the rule by which he chooses his strategy at time t , knowing his objective function and admissible set, knowing (and sometimes not knowing or only partially knowing) the objective functions and admissible sets of other elements and knowing (and sometimes not knowing or knowing only partially) the history of the game $h(\tau)$:

$$(8.1) \quad s_i(t) = F_i(\Omega, \varphi, h(t), t), \quad i = \overline{1, n}, \quad t > 0.$$

Anticipating possible objections to vesting elements of the learning system with some “interests”, we note that, indeed, in active systems (for example, a group of interacting operators), the functions $\{\varphi_i, F_i\}$ reflect the interests of system elements, and in passive systems $F_i(\cdot)$ is nothing more than a law (sometimes unknown to the researcher) of the change of state of elements that satisfies physical, biological and other restrictions.

It is clear that by adopting one hypothesis or another about the behavior of elements and their interaction, it is possible to calculate the trajectories of each of them. With an increase in the dimensionality of the system, the expediency of using this method becomes problematic and there is a desire to describe the behavior of the system as a whole (it may be somewhat averaged and not quite accurate) without going into a detailed description of each of the elements.

Intuitively, in some cases such an aggregated description will turn out to be more accurate with an increase in the dimension of the system.

In the particular case (8.1) turns into a dynamical system

$$(8.2) \dot{s}_i = f_i(s(t)), i = \overline{1, n}, t > 0,$$

or, if time is discrete, (8/1) turns into a system of difference equations:

$$(8.3) s_i(k + 1) = f_i(s(k)), i = \overline{1, n}, k = 0, 1, 2, \dots$$

In the last two cases, the task of studying the dynamics of collective behavior is reduced to studying the properties of a dynamic system [51]. In particular, it is necessary to determine whether there exists an equilibrium point (sometimes this is equivalent to studying the existence of the equilibrium position of a dynamic system) and whether it is stable, whether the trajectories of the system converge to this equilibrium position (what are the regions of attraction of various equilibrium points), what is the rate of convergence and so on. To date, answers to these questions in the general case do not exist, and most studies have concentrated on the study of particular private models.

Model 8.4.

D(H). The states of the elements of the system satisfy the normal system of differential equations:

$$(8.4) \dot{s}_i = f_i(s(t), t), i = \overline{1, n}, t > 0.$$

Suppose that the functions $\{f_i\}$ are continuous and Lipschitz (satisfying a certain restriction on the growth rate) in the entire admissible region.

T(C). For any admissible initial point, a solution to system (8.4) exists and is unique. Moreover, if the solution (8.4) is asymptotically stable, then the equilibrium position is reachable in infinite time (group property).

If $\{f_i\}$ are linear functions and all eigenvalues of the corresponding matrix have negative real parts, then there are two exponential functions that limit the trajectory of system (8.4) from above and below. The introduction of an additional assumption about the monotonicity of the right-hand side of system (8.4) leads to the slow-asymptotic form of the trajectories of its solution.

A. The Lipschitz character of the right-hand side of the system of differential equations can be interpreted as the limitation of the rate of possible changes in the states of the elements (and, consequently, the mismatch), leading to the unattainability of the equilibrium position (zero error) for a finite time. In order to exclude the possibility of inflection points, one should introduce a sufficiently strong assumption about the monotonicity of the right-hand side. •

One of the most common and well-studied assumptions about the rational behavior of elements of any active system is the hypothesis of indicator behavior. In accordance with this hypothesis, at each iteration, each element takes a step in the direction of the strategy that would be optimal if all other elements would choose the same strategies as in the previous step. In this case, we determine the position of the target of the i -th element:

$$w_i(s_{-i}) = \arg \max_{s_i \in \Omega_i} \varphi_i(s_i, s_{-i}),$$

where $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$ – is the game situation for i -th element.

Then the hypothesis of indicator behavior can be written as

$$s_i(k+1) = s_i(k) + \gamma_i^k (w_i(s_{-i}(k)) - s_i(k)), \quad i = \overline{1, n}, \quad k = 0, 1, 2, \dots,$$

where parameters $0 \leq \gamma_i^k \leq 1$ define "step sizes". A detailed study of systems in which elements behave in accordance with the indicator behavior hypothesis was carried out in [42, 51].

With an increase in the number of elements with "approximately the same" effect on the system as a whole, it turns out that the behavior of the system is determined by some "averaged" element. At the same time, there is no need to study all the elements — the values of the indicators characterizing the entire system turn out to be stable over a fairly wide range of element parameter values [52, 53, 54]. The possibility of such an "averaging" (without significantly losing the accuracy of the description) seems quite attractive, since the number of elements in real iteratively taught systems is usually extremely large (it doesn't matter at all what to mean by an "element" as a brain neuron, degree of freedom of the hand etc.). An example of the use of asymptotic aggregation methods in the study of collective behavior (within the framework of the indicator behavior hypothesis) is the model below (a reader who is not familiar with the apparatus used may skip the following formal results whose boundaries are marked with " \square ").

Model 8.5.

D. Consider a system consisting of n interconnected elements that function in discrete time. The state of the system at time k : $s^k = (s_1^k, s_2^k, \dots, s_n^k) \in \Omega \subseteq \mathcal{H}^n$ is determined by the states of the $s_i^k \in \Omega_i$, $k = 1, 2, \dots$, where

$$\infty < s_i^- < s_i^+ < +\infty, \quad i = \overline{1, n}.$$

H. Suppose that the behavior of the system satisfies the hypothesis of indicator behavior - at each moment in time, each of the elements changes its state in the direction of the current position of the target, i.e. described by an iterative type procedure

$$(8.5) \quad s_i^{k+1} = s_i^k + \gamma_i^k [w_i(s_{-i}^k) - s_i^k], \quad k = 1, 2, \dots, i = \overline{1, n}.$$

where $w_i(s_{-i}^k)$ - the current position of the target of the i -th element, which depends on the states of the remaining elements, and the parameters $\gamma^k = (\gamma_1^k, \gamma_2^k, \dots, \gamma_n^k)$, chosen by the elements, determine the values of steps (learning speed) and have arbitrary distributions in a unit cube.

□ Suppose that the equilibrium point of the system $c = (c_1, c_2, \dots, c_n)$, $c_i \in [s_i^-; s_i^+]$, $i = \overline{1, n}$, exists, is unique, and the trajectories (8.5) converge to this point (the corresponding conditions are given, for example, in [42, 51]).

As a measure of the current "remoteness" of the system from the equilibrium position, we choose the mismatch

$$(8.6) \quad \Delta_n^k = \|c - s^k\| = \frac{1}{n} \sum_{i=1}^n |c_i - s_i^k|,$$

i.e. the distance between points s and c in \mathcal{R}^n space.

T. Using (8.5), we obtain:

$$(8.7) \quad \Delta_n^{k+1} = \frac{1}{n} \sum_{i=1}^n |(c_i - s_i^k)(1 - \gamma_i^k) + \gamma_i^k (c_i - w_i(s^k))|.$$

Obviously: $\Delta_n^{k+1} \leq \tilde{\Delta}_n^{k+1}$, where

$$(8.8) \quad \tilde{\Delta}_n^{k+1} = \frac{1}{n} \sum_{i=1}^n |c_i - s_i^k| (1 - \gamma_i^k) + \frac{1}{n} \sum_{i=1}^n \gamma_i^k |c_i - w_i(s^k)|.$$

For sufficiently large n , the mismatch estimate $\tilde{\Delta}_n^{k+1}$ should differ slightly from the "average value"

$$(8.9) \quad \bar{\Delta}_n^{k+1} = (1 - \bar{\gamma}_n^k) \Delta_n^k + \bar{\gamma}_n^k \frac{1}{n} \sum_{i=1}^n |c_i - w_i(s^k)|,$$

where $\bar{\gamma}_n^k = \frac{1}{n} \sum_{i=1}^n \gamma_i^k$.

Let's give the correct formulation and justification of this statement. We define what is meant by proximity $\tilde{\Delta}_n^{k+1}$ and $\bar{\Delta}_n^{k+1}$. According to [53, 54], the sequence of functions $\tilde{\Delta}_n^{k+1}(\gamma^k)$ stabilizes on unit cubes $K_n = [0; 1]^n$ if there exists a numerical sequence $\bar{\Delta}_n^{k+1}$ such that

$$(8.10) \Pr \{ |\tilde{\Delta}_n^{k+1} - \bar{\Delta}_n^{k+1}| \geq \varepsilon \} \rightarrow 0, n \rightarrow +\infty$$

for any given $\varepsilon > 0$.

In order to conclude something about the stabilization, we estimate the difference in the values of the function $\tilde{\Delta}_n^{k+1}(\cdot)$ in the following points:

$$\gamma^k \in K_n \text{ и } \delta^k = (\delta_1^k, \delta_2^k, \dots, \delta_n^k) \in K_n:$$

$$\begin{aligned} |\tilde{\Delta}_n^k(\gamma^k) - \tilde{\Delta}_n^k(\delta^k)| &= \left| \frac{1}{n} \sum_{i=1}^n |c_i - s_i^k| (\delta_i^k - \gamma_i^k) + \right. \\ &\quad \left. + \frac{1}{n} \sum_{i=1}^n (\gamma_i^k - \delta_i^k) |c_i - w_i(s^k)| \right|. \end{aligned}$$

Denoting $\alpha = \max_i (s_i^+ - s_i^-)$, obtain

$$|\tilde{\Delta}_n^k(\gamma^k) - \tilde{\Delta}_n^k(\delta^k)| \leq \frac{2\alpha}{n} \sum_{i=1}^n |\gamma_i^k - \delta_i^k|,$$

so $\tilde{\Delta}_n^k(\cdot)$ is a Lipschitz function with Lipschitz constant of order $1/n$.

By virtue of Theorem 2 [54], for any distributions γ^k on K_n dispersion $D\{\tilde{\Delta}_n^k\} \rightarrow 0, n \rightarrow +\infty$, therefore, by the Chebyshev inequality, (8.10) holds.

C. The stabilization of sequence $\tilde{\Delta}_n^k$ allows us to formulate the following conclusion. With an increase in the number of system elements, estimate (8.8) of the mismatch (8.6) converges in probability to (8.9), i.e. occurs:

$$(8.11) \Pr \{ \Delta_n^{k+1} > (I - \bar{\gamma}_n^k) \Delta_n^k + \bar{\gamma}_n^k \frac{1}{n} \sum_{i=1}^n |c_i - w_i(s^k)| \} \xrightarrow{n \rightarrow +\infty} 0.$$

Some special cases of the above statement are considered below:

- if the system moves monotonously to the equilibrium position (if $s_i^k \geq c_i$, then $s_i^k \geq w_i(s^k) \geq c_i$ and, accordingly, if $s_i^k \leq c_i$, then

$s_i^k \leq w_i(s^k) \leq c_i, i = \overline{1, n}, k = 1, 2, \dots$, then (8.7) converges in probability to (8.9)

- if the system moves monotonously to the equilibrium position (if, then, and accordingly, if), then (8.7) converges in probability to (8.9);

- if the elements of the system do not interact or there exist $\delta > 0: |c_i - w_i(s^k)| \sim o(n^\delta), i = \overline{1, n}, k = 1, 2, \dots$, then (8.7) converges in probability to $(1 - \bar{\gamma}_n^k) \Delta_n^k$. \square

A. The study of the model allows us to make the following qualitative conclusion: if

- elements do not interact, or

- the position of the target does not change over time (for example, $w_i = c_i$), or

- the average change in the position of the target relative to the equilibrium points for each element at each step is quite small:

$$|c_i - w_i(s^k)| \ll |c_i - s_i^k| \quad \forall i = \overline{1, n}, k = 1, 2, \dots ,$$

then the average mismatch can be accurately approximated by an exponential curve.

The assumption about the validity of the indicator behavior hypothesis and the choice of the mismatch in the form (8.6) are essential in this model. Moreover, the assumption of the stationary position of the target, in fact, reduces the model in question to model 4.1. •

In collective behavior models, the slow-asymptotic nature of the LC is a consequence of either a large number of elements of the system, or / and the absence or limitation of their interaction, or / and the constancy of the position of the goal.

9. Some generalizations

As has been repeatedly noted above, iterative learning is characterized by the constancy of external conditions and learning objectives, that is, there is a stationarity of external (with respect to the learning system) parameters (functioning conditions). We show that in order to explain the slow-asymptotic (exponential) nature of the iterative learning curves, it suffices to introduce the assumption that some parameters of the learning system itself (internal operating conditions) are stationary. Moreover, this assumption is sufficient to explain a much wider range of phenomena and processes than just IL - starting from a number of physical and chemical laws and ending with the processes of self-organization and adaptation in complex biological and cybernetic systems.

Consider the following model, which is a generalization of almost all the models considered above in the following sense: we will not introduce assumptions about the nature, laws, etc. the interaction of elements and the structure of the system, assuming that there are some characteristics of the elements (their mismatch) that determine the mismatch of the system.

D. Consider a system consisting of n elements. The discrepancy of the i -th element is denoted by $x_i(t)$, $i = \overline{1, n}$. Without loss of generality, we can assume that if the system learns, then there is

$$x_i(0) = 1, x_i(t) > 0 \quad \forall t > 0, \lim_{t \rightarrow \infty} x_i(t) = 0, i = \overline{1, n}.$$

Any curve of this type can be represented as

$$(9.1) \quad x_i(t) = e^{-\xi_i(t)}, \quad i = \overline{1, n}.$$

where $\xi_i(0) = 0$, $\xi_i(t) \xrightarrow{t \rightarrow \infty} +\infty$, $i = \overline{1, n}$. We will conventionally call the learning speed of the i -th element the logarithmic derivative of its mismatch ("relative speed" \dot{x}_i / x_i), that is, the quantity $\gamma_i(t) = \frac{d\xi_i(t)}{dt}$ (in each case, you need to clearly understand what is an element of the simulated system and what are the meaningful interpretations of the speed of the learning).

As a rule, the trajectories of real physical and biological systems are sufficiently smooth, therefore, in most cases, the corresponding derivative is defined. If $\xi_i(\cdot)$ are absolutely continuous functions (which, up to a

constant, can be represented as an integral of the derivative), then (9.1) takes the form

$$(9.2) \quad x_i(t) = \exp \left\{ - \int_0^t \gamma_i(\tau) d\tau \right\}, \quad i = \overline{1, n}.$$

H. The mismatch of the system as a whole is some function of the mismatch of the elements:

$$x(t) = F(x_1(t), x_2(t) \dots, x_n(t)).$$

It is natural to assume that the function $F(\cdot)$ is non-negative, monotonic in each variable and vanishes if and only if the mismatches of all elements vanish. For example, $F(\cdot)$ may be the norm in the space \mathcal{R}^n . It is known that in finite-dimensional spaces (in the model under consideration, the dimension of space is determined by the number of elements of the system being studied, and it is always finite), all norms are equivalent, that is, for any two norms $F_1(\cdot)$ and $F_2(\cdot)$ there exist constants α and β , such that for any $x \in \mathcal{R}^n$

$$(9.3) \quad \alpha F_2(x) \leq F(x) \leq \beta F_1(x).$$

Let the mismatch $F(\cdot)$ be the geometric mean of the mismatch of elements:

$$(9.4) \quad x(t) = \left[\prod_{i=1}^n x_i(t) \right]^{1/n} = \exp \left\{ - \int_0^t \frac{1}{n} \sum_{i=1}^n \gamma_i(\tau) d\tau \right\}.$$

If you choose the arithmetic mean of the mismatch of the elements mismatch:

$$F(x_1(t), x_2(t) \dots, x_n(t)) = \sum_{i=1}^n |x_i(t)|,$$

then for sufficiently large n the arithmetic mean “coincides” (accurate to the multiplicative constant) with the geometric mean (the correct justification is given in [53]).

Thus, in order for (9.4) to be, in the sense defined in [53], an estimate of the mismatch of the system (see (9.3)), it is necessary that the number of elements of the system be large.

T(C). Now we use the hypothesis of stationarity of the characteristics of elements. More precisely, we assume that the element learning rates are independent random variables having arbitrary stationary distributions.

Then the integrand in (9.3) is asymptotically constant [54], that is, for large n :

$$(9.5) \quad \frac{1}{n} \sum_{i=1}^n \gamma_i(t) \approx \text{Const}, \quad \forall t \geq 0.$$

Denoting this constant (learning rate) by γ , from (9.4) and (9.5) we obtain:

$$(9.6) \quad x(t) \approx e^{-\gamma t}.$$

A. Thus, within the framework of the considered model, the exponential form of the time dependence of the system mismatch is a consequence of the stationary nature of the external and internal parameters (operating conditions), as well as a large number of system elements.

The presence of a large number of elements of the system is essential - the "learning curves" of individual elements can be far from exponential. Roughly speaking, the greater the number of elements of the system and the more "stationary" their characteristics, the more accurately (9.6) approximates the learning curve of the system.

It should be noted that the proposed model is far from perfect. For example, an attentive reader may ask: why did we use representation (9.1) for the "learning curve" of an individual element? (assuming the derivatives $\dot{x}_i(t)$ are stationary, then we get a linear function that does not satisfy the asymptotic condition), what is the function $\xi_i(t)$, and why is the distribution of its derivatives stationary? Similar objections may cause the validity of assumptions about the properties of the function $F(\cdot)$, the independence of the characteristics of the elements, etc.

The following reasoning may serve as a justification. Let a certain system be characterized by an exponential LC with a learning speed whose value γ , in fact, determines the difference between one LC (learning system) and another. In constructing the IL model, the researcher, when considering the interaction of system elements, is forced to introduce certain assumptions. As the analysis above shows, there are a number of assumptions that lead to the required conclusion about the exponential dynamics of the behavior of the system. Therefore, the introduced assumptions should serve as a criterion for comparing IL models (which model is "better"). From this point of view, the proposed model is "better" than those considered above (it is more general, that is, it includes most of the known models as special cases). The process of generating models can and should continue. Nevertheless, one must clearly realize that it is most likely that it will be impossible to completely abandon the assumptions about the sta-

tionarity and / or boundedness of certain system parameters and / or the multiplicity of its constituent elements. •

Now we show that the presented model generalizes the models considered in the previous sections.

A large number of elements of the learning system is essential in the models: 4.2, 5.2, 5.3, 6.1, 6.7, 6.10, 7.2, 7.5, 8.5.

Stationarity of characteristics of system elements takes place and plays a key role:

- in models 4.1, 4.5, 5.1, 5.4, 6.5, 8.2, 8.3 the logarithmic derivative of the mismatch is constant, that is, the proportionality coefficient between the rate of change of mismatch and its current value is constant (it is clear that even with $n = 1$ this assumption is immediate leads to the exponential form of the mismatch curve);

- in models 4.2, 7.3, 7.4, the proportionality coefficients in the expressions for the increment of probabilities are constant;

- in model 5.2, the probability of "decay" does not depend on the time and number of "decaying atoms";

- in model 5.3, the relative error of each of the regulators is constant;

- in model 5.5, the variation in the organization of the system is constant (equal to zero);

- in models 6.2, 6.3, 6.6, the amount of information absorbed, received or processed by the trained system per unit of time is constant;

- in model 6.7, the likelihood of displaying various images is equal;

- in model 6.9, the proportion between the transmitted and received information does not depend on the time and amount of accumulated information;

- Model 6.10 (like 5.2) is very close to the model discussed in this section;

- in model 7.1, the proportion of split segments is constant;

- in model 7.2, the mismatches of the elements of the system are distributed evenly at each moment of time;

- in model 8.4, the limited rate of change (Lipschitz) of the right-hand sides of the normal system of differential equations is sufficient for the asymptotic behavior of the system trajectory;

- in model 8.5, the non-interaction of elements or the constancy of the position of the target lead to the exponential form of the mismatch curve.

At the same time, it should be noted that during iterative learning in the case of unsteady internal characteristics of the system, non-exponential (logistic, with an intermediate plateau, etc. - see the second section) learning curves can be observed and repeatedly observed in experiments.

Thus, in the above models, to obtain a conclusion on the exponentiality of the learning curve, either assumptions are made about the multiplicity and uniformity of system elements (see also the remark on the need to average individual LCs in the second section) and the stationarity of some characteristics of elements (multiplicity for “weak” stationarity makes it possible to “average” and obtain “strong” stationarity “on average”; intuitively, “small non-stationarity” leads to “approximate exponentiality”), or a stronger assumption of stationarity.

So we can draw the following conclusion: if the number of elements of the system being taught is large enough, and its characteristics and functioning conditions (internal and external) are stationary, then the corresponding learning curve will be exponential. Moreover, the model presented in this section turns out to be adequate not only for iterative learning, but also for the processes of self-organization and adaptation in large systems that satisfy stationarity assumptions.

Conclusion

Thus, the analysis of mathematical models of iterative learning carried out in this work allows us to draw the following conclusions.

Modeling iteratively learned systems is an effective method of their study, predicting the specifics of the behavior of real systems in various conditions, as well as improving the organization of the educational process.

The results of the study of mathematical models of iterative learning allowed us to put forward the following law of iterative learning:

IF THE NUMBER OF ELEMENTS OF THE LEARNING SYSTEM IS SUFFICIENTLY GREAT AND / OR EXTERNAL AND INTERNAL CONDITIONS OF ITS FUNCTIONING ARE STATIONARY, THEN THE LEARNING CURVE IS EXPONENTIAL.

At the same time, a dual statement can be formulated (put forward as an explanatory hypothesis):

IF THE LEARNING CURVE IS EXPONENTIAL, then, probably, THE EXTERNAL AND INTERNAL CONDITIONS OF ARE STATIONARY THE NUMBER OF ELEMENTS OF THE LEARNING SYSTEM IS SUFFICIENTLY GREAT.

The formulated statements are quite consistent with the corresponding experimental laws, physical laws, and observation results.

As promising areas for future research on the mechanisms and patterns of iterative learning, it is worth highlighting: the need for further analysis of various types of IN models and, first of all, models using the inverse construction method; the study of the correspondence between the hypotheses underlying the existing and newly created direct models of IN and experimental studies of IN in living systems; as well as the widespread use of simulation results to develop recommendations for the selection of optimal forms and methods of training.

It should be noted that many of the models considered above describe and reflect a much wider range of phenomena and processes than just iterative learning. It can be hypothesized that the slow-asymptotic nature of the change in the aggregated parameters of large and complex systems is a general pattern that manifests itself under stationary external and internal conditions not only during iterative learning, but also in the processes of adaptation, self-organization, etc.

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