"FAIR PLAY" IN CONTROL OF ACTIVE SYSTEMS

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Large-scale manmachine systems incorporate subsystems whose goals do not generally coincide with those of the system. A single man or a group of people make a subsystem actively maximize its objective function by reporting the information on its model (in other words on its potential) to an external control unit. Besides the subsystem has certain information on the strategy applied by the external control unit and by other subsystems and uses this information in itw own interests. This lecture is concerned with control of such active systems that incorporate men and groups of people that are after their own goals. The control problem is to find an optimal plan for the system so that the subsystems plans be also optimal. A solution of the control problem based on the "fair play" principle is proposed. This principle largely recognizes the active nature of the subsystems.

1. Active Systems

A multi-level system is defined if,

(a) *The system structure is defined,* i.e. for each subsystem (SS) the controlling (master) subsystem and a set of controlled (slave) subsystems are known;

(b) *the model of each subsystem is known,* i.e., the way to represent a set of possible plans is defined and the objective function for each subsystem is given which depends on the plan for the given subsystem, on the plans for its "slave" subsystems, a control generated by the controlling subsystem and a control generated by the given subsystem for the controlled sub-subsystems;

(c) *the relations of plans for the lower and upper level subsystems are determined;* i.e. a certain plan Z^i (W^i) for the subsystem i is associated with each set $W^i = (W^{i1}, W^{i2}, W^{is})$ of feasible plans for the "slave" subsystems. The number of parameters which describe the plan Z^i is less than the number describing the set of plans W^i, so the information aggregation takes place. The controlling subsystem of the first hierarchical level is hereafter termed the central unit

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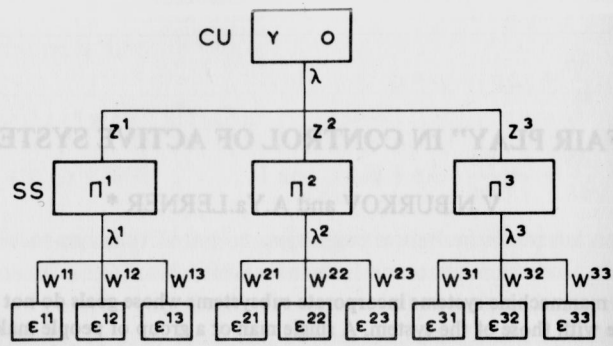

Fig. 1.

(CU) while subsystems which have no slave subsystems are called elements (E).

Example 1. Fig. 1 shows a structure of a three-level system of nine elements.

Assume that the set A^{ij} of feasible plans for each element is described by the inequality

$$\sum_{k=1}^{n} W_k^{ij} b_k^{ij} \leqslant T \,, \tag{1}$$

where b_k^{ij} and T are positive numbers, W_k^{ij} are non-negative numbers. Such a model can describe a factory which manufactures n kinds of products for a period T. Then b_k^{ij} is the time spent on manufacturing a unit of the k-th product W_k^{ij} is a plan for this type of product.

The objective function of an element may have the form

$$\eta^{ij}(W^{ij}) = \sum_{k=1}^{n} \lambda_k^i W_k^{ij} \,,$$

where λ^i is the control generated by the i-th subsystem (the vector of product prices). In this case the set of feasible plans is determined by setting n parameters b_k^{ij}. The i-th subsystem (a particular industry) plan is

$$Z^i = \sum_{j=1}^{s} W^{ij} \,,$$

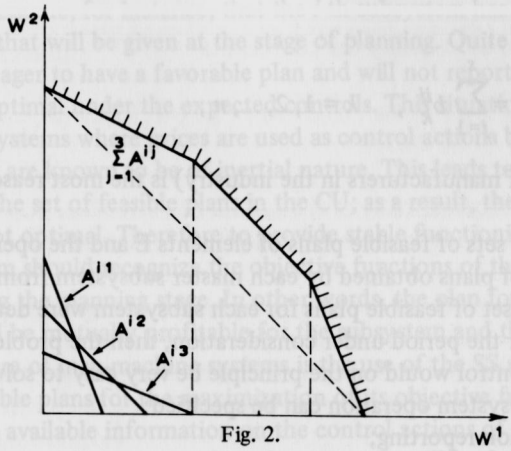

Fig. 2.

(total of all kinds of products made by all manufacturers of the industry). In this case a set of possible plans is formed as (see fig. 2)

$$\sum_{j=1}^{s} A^{ij} .$$

Let the set A^i of feasible plans for the i-th industry be described by the following inequality

$$\sum_{k=1}^{n} Z_k^i b_k^i \leqslant T , \tag{2}$$

which is similar to (1).

In other words, the industry acts as a single large manufacturer. Since the set

$$\sum_{j=1}^{s} A^{ij}$$

cannot be accurately described in terms of (2), a set of feasible plans has to be represented approximately. For instance if each plan of the set A^i is to be feasible, that is

$$A^i \subseteq \sum_{j=1}^{s} A^{ij} ,$$

then,

$$b_k^i = \sum_{j=1}^{s} b_k^{ij} \ , \qquad k = 1, 2, ..., n \ ,$$

(the number of manufacturers in the industry) is the most reasonable representation.

Note that if sets of feasible plans of elements E and the operators transforming a set of plans obtained by each master subsystem from its slave subsystems into a set of feasible plans for each subsystem were determined single-meaningly over the period under consideration, then the problem of a multi-level system control would on the principle be very easy to solve. Indeed, three stages of system operation can be specified:

a) information reporting;

b) planning;

c) plan execution.

At the first stage each E reports to its master (*SS*) a set of possible plans. Having obtained a set of feasible plans from the slave subsystems, each subsystem finds a set of its own feasible plans and passes them over to the corresponding master subsystem. With all the information received the CU solves the planning problem (the stage of planning) finds the control λ and the 2nd level subsystems plans maximizing (or minimizing) the objective function of the system. Having received the plan Z^i the i-th subsystem finds in its turn the control λ^i and the plans W^{ij} for the slave subsystems so as to maximize its own objective function, etc. Finally, at the stage of plan execution each E fulfills the given plan. Knowing the states of plan-fulfillment for the elements we can exactly determine the states of plan-fulfilment for the subsystems and for the system.

Note that in the model described above each SS is passive at the stage of information reporting. It only transforms and reports information to other SS. However this assumption is invalid for man-machine systems. It would be more realistic to assume that while reporting the information on the set of its feasible plans each subsystem acts in its own interests (of course, within the established forms of information presentation and constraints). Then each SS will naturally report only "favorable" plans.

Therefore plans that cannot be optimal under any feasible control are never reported. The situation is more complicated for plans that can be optimal for a subsystem under some controls. The point is that in the described model no subsystem takes into account the objective functions of its slave SS while solving the planning problem. This causes a certain instability in the system

functioning. Assume, for instance, that the i-th subsystem has information on the control λ that will be given at the stage of planning. Quite naturally the subsystem is eager to have a favorable plan and will not report those plans that are not optimal under the expected controls. This situation can well arise in economic systems where prices are used as control actions because the variations in prices are known to be of inertial nature. This leads to a substantial reduction of the set of feasible plans in the CU; as a result, the obtainable plan is generally not optimal. Therefore to provide stable functioning of the system each subsystem should recognize the objective functions of the slave subsystems during the planning stage. In other words, the plan found by the subsystem should be mutually profitable for the subsystem and the slave SS. Thus, a salient feature of man-machine systems is the use of the SS reports on the set of its feasible plans for the maximization of its objective function and also the sue of the available information on the control actions of the master SS the CU including. Subsystems with such a property will be further referred to as active subsystems (ASS) and a system with at least one ASS (apart from the CU) – an active system (AS). It has already been stated that a necessary condition for the stability of AS's is the mutual profitability of plans.

2. Problem Statement

In further discussion we will deal with two-level systems consisting of a CU and a finite number of active elements (AE_s). The results, however, can be easily extended to multi-level systems. Let us introduce the following notation.

A^i is a set of feasible plans for the i-th AE.

$\hat{Z}^i$ is the plan for the i-th AE obtained from the CU at the planning stage.

$\hat{Z} = (\hat{Z}^1, \hat{Z}^2, ..., \hat{Z}^m)$

is a plan for an AS (a set of AE's plan;

λ is the control vector;

L is the set of feasible controls;

$\mathcal{Z}$ is a set of feasible plans for the AS.

$\eta^i = (Z^i, \lambda)$

is the objective function of the i-th AE.

$\Phi(Z, \lambda)$ is the objective function of the AS.

Each AE knows a set A^i and its own objective function $\eta^i(Z^i, \lambda)$.

The CU in its turn knows the objective function of the system, a set of feasible controls L and the constraints $\mathcal{Z}$ on the plans of various AE's. Note that this information may also be known to each AE.

Problem formulation. Find the plan $\hat{Z} \in \mathcal{Z}$, $\hat{Z}^i \in A^i$, $i = 1, 2, ..., m$ and the control $\lambda \in L$ so as to satisfy the constraints

$$\eta^i(\hat{Z}^i, \lambda) = \max_{Z^i \in A^i} \eta^i(Z^i, \lambda), \qquad i = 1, 2, ..., m \tag{3}$$

and obtain the maximal value of the system objective function $\Phi(\hat{Z}, \lambda)$. Conditions (3) + $\Phi(Z, \lambda) \to \max$ are the conditions for mutual profitability of the plan $\hat{Z}$.

Since the conditions for mutual profitability of the plan put additional constraints on the set of feasible plans, the value of the objective function given by the mutually favorable plan will generally be below the one given by the optimal plan without the requirement for the mutual profitability.

Denote:

Φ_0 is the maximal value of the AS objective function without the requirement for the mutual profitability;

Φ_b is the value of the AS objective function given by the optimal mutually profitable plan. (Without loosing the generality assume that $\Phi_0, \Phi_b > 0$.)

The relation

$$\frac{\Phi_b}{\Phi_0} = \rho$$

is termed the AS *matching coefficient.* It shows the degree of matching between the objectives of the AS and the objective of the entire system. If $\rho = 1$ then the AS is completely matched.

Let $Z(\lambda)$ denote the optimal solution to the planning problem under the control $\lambda \in L$. The problem reduces to finding the control $\lambda \in L$ under which $\Phi[Z(\lambda), \lambda]$ takes the maximal value.

Example 2. Consider the problem of planning for two manufacturers who can make two types of products. Assume that the period T is equal to 1. Let one manufacturer have the production rate of any type of product equal to 1, while the other one have the production rate of the first type product equal to 1, and of the second equal to ½. Let Z_j^i be the time spent by the i-th manufacturer on the production of the j-th product type ($i, j = 1, 2$) λ_1, the relative profit secured by the production of first product type, λ_2- of second product type, $\lambda_1 + \lambda_2 = 1$.

We have a set of feasible plans for the first manufacturer

$$A^1: Z_1^1 + Z_2^1 \leqslant 1, \qquad Z_1^1, Z_2^1 \geqslant 0,$$

the objective function (the profit defined within the accuracy to a constant multiplier) is equal to

$$\eta^1(Z^1, \lambda) = \lambda_1 Z_1^1 + \lambda_2 Z_2^1 ;$$

a set of feasible plans for the second manufacturer

$$A^2 : Z_1^2 + Z_2^2 \leqslant 1 , \quad Z_1^2, Z_2^2 \geqslant 0 ;$$

the second manufacturer objective function

$$\eta^2(Z^2, \lambda) = \lambda_1 Z_1^2 + \tfrac{1}{2}\lambda_2 Z_2^2 ;$$

the set of feasible controls

$$L : \lambda_1 + \lambda_2 = 1 ; \quad \lambda_1, \lambda_2 \geqslant 0 .$$

Suppose the AS objective function has the form:

$$\Phi(Z, \lambda) = 4Z_1^1 + 2Z_2^1 + Z_1^2 + 2Z_2^2 .$$

Determine the function $\Phi[Z(\lambda), \lambda]$

(a) $0 \leqslant \lambda_2 \leqslant \frac{1}{2}$.

The optimal mutually profitable plan is

$$Z_1^1 = 1, \quad Z_2^1 = 0 , \quad Z_1^2 = 1 , \quad Z_2^2 = 0 ; \quad \Phi[Z(\lambda), \lambda] = 5 .$$

(b) $\frac{1}{2} < \lambda_2 < \frac{2}{3}$.

The mutually profitable plan is

$$Z_1^1 = 0 , \quad Z_2^1 = 1 , \quad Z_1^2 = 1 , \quad Z_2^2 = 0 ; \quad \Phi[Z(\lambda), \lambda] = 3 .$$

(c) $\frac{2}{3} \leqslant \lambda_2 \leqslant 1$.

The mutually optimal plan is

$$Z_1^1 = 0, \quad Z_2^1 = 1 , \quad Z_1^2 = 0 , \quad Z_2^2 = 1 ; \quad \Phi[Z(\lambda), \lambda] = 4 .$$

Note that the function $\Phi[Z(\lambda), \lambda]$ is a multi-extremum function of λ. The optimal plan without the mutual profitability requirement

$$Z_1^1 = 1, \quad Z_2^1 = 0, \quad Z_1^2 = 0, \quad Z_2^2 = 1; \quad \Phi_0 = 6, \quad \Phi_b = 5.$$

The matching coefficient of the system is $\rho = \frac{5}{6}$.

Example 3. The data are the same as in Example 1, but the objective function of the system is

$$\Phi(Z, \lambda) = \min(4Z_1^1 + Z_1^2; 2Z_2^1 + 2Z_2^2).$$

(a) $0 \leqslant \lambda_2 < \frac{1}{2}$.

The optimal mutually profitable plan is

$$Z_1^1 = 1, \quad Z_2^1 = 0, \quad Z_1^2 = 1; \quad Z_2^2 = 0; \quad \Phi[Z(\lambda), \lambda] = 0.$$

(b) $\lambda_2 = \frac{1}{2}$, $\lambda_1 = \frac{1}{2}$.

The optimal mutually profitable plan is

$$Z_1^1 = \tfrac{1}{6}, \quad Z_2^1 = \tfrac{5}{6}, \quad Z_1^2 = 1, \quad Z_2^2 = 0, \quad \Phi[Z(\lambda), \lambda] = 1\tfrac{2}{3}.$$

(c) $\frac{1}{2} < \lambda_2 \leqslant \frac{2}{3}$.

The optimal mutually profitable plan is

$$Z_1^1 = 0, \quad Z_2^1 = 1, \quad Z_1^2 = 1, \quad Z_2^2 = 0; \quad \Phi[Z(\lambda), \lambda] = 1.$$

(d) $\frac{2}{3} < \lambda_2 \leqslant 1$.

The optimal mutually profitable plan is

$$Z_1^1 = 0, \quad Z_2^1 = 1, \quad Z_1^2 = 0, \quad Z_2^2 = 1; \quad \Phi[Z(\lambda), \lambda] = 0.$$

In this example the optimal value $\Phi_b = 1\frac{2}{3}$ is achieved at the point $\lambda_1 = \lambda_2 = \frac{1}{2}$ where the function $\Phi[Z(\lambda), \lambda]$ has no derivative.

The optimal plan without the mutual profitability requirement is then

$$Z_1^1 = \tfrac{2}{3}, \quad Z_2^1 = \tfrac{1}{3}, \quad Z_1^2 = 0, \quad Z_2^2 = 1; \quad \Phi_0 = 2\tfrac{2}{3}.$$

The matching coefficient in this example is $\rho = \frac{5}{8}$.

The basic difficulty in solving the planning problem is as already noted in that the CU either does not know the sets A^i and theobjective functions $\eta^i(Z^i, \lambda)$ of the active elements or know them only approximately. Therefore a certain procedure must be devised whereby the CU will receive the information on the sets of AE's mutually profitable plans.

One approach developed by several authors which is currently used to solve the planning problem is the principle of decomposition. The basic idea is to arrange an iterative procedure. At each iteration the CU reports to the active elements the control λ_k while each AE reports the plan Z_k^i under this control (k is the number of iteration). The control λ_{k+1} at the $(k+1)$-th iteration is determined by a certain law $\lambda_{k+1} = \Psi_{k+1}(Z_k, \lambda_k)$ in terms of the control λ_k and the plan Z_k at the preceding iteration.

The CU should evidently find the conditions at which the procedure ends (conditions of stoppage).

If the procedure ends at the s-th iteration, then $\lambda = \lambda_s$, $Z = Z_s$ is taken as the solution to the planning problem.

In the model under consideration the information reporting and planning stages are integrated. Now, since they are active, AS's may know the stoppage condition and thus predict the last iteration.

To continue the discussion we need to assume that each AE has a slight effect on the stoppage conditions. This assumption can be formulated as follows: for any $Z_k^i \in A^i$ there is a probability not equal zero that the k-th iteration is the last one for the i-th AE, ($i = 1, 2, ..., m$, $k = 1, 2, ...$). If to assume now that each AE maximizes the guaranteed value of the objective function, then with the above assumption valid the E will report at each iteration one of the Z_k^i that maximize $\eta^i(Z_k^i, \lambda_k)$. A number of papers [1–5] prove the convergence of the sequence $\{\lambda_k, Z_k\}$ to an optimal solution when there are certain constraints on the properties of the sets A^i, $\mathcal{Z}$, L and the functions $\eta^i(Z^i, \lambda)$ and $\Phi[Z, \lambda]$. The approach considered can easily be extended to multi-level systems. Sometimes more complicated models are used where at each iteration each AE reports to the master SS several close plans with the appropriate controls [6]. Without going into details of these methods which use the decomposition principle, note that they do not give the optimal solution since the problem is multi-extremal. (See Examples 1 and 2.) Besides, there can also be several optimal AE's plans at the control selected and it is not clear which one will be reported to the CU. This fact has already been demonstrated for linear models [7]:

"Even for simple economic systems represented in terms of linear programming it is impossible to construct a local function (even using the estimates of the global problem optimal plan) so as to make a solution to the local problem optimal to the global problem".

A new approach to the solution of AS control problem based on *the "fair play" principle* will be discussed below.

3. The Principle of Fair Play in Control

According to the principle of fair play each AE reports a set of feasible plans B^i and the preference function $S^i(Z^i, \lambda)$ defined on that set to the CU which solves the planning problem on the knowledge of this information using the preference function as the objective function. Assume $S^i(Z^i, \lambda) = -\infty$ for $Z_i \notin B^i$. Each AE can now be considered to have reported only the preference function on the given set A^i.

Condition (3) for mutually profitable planning will in this case take the form

$$(4) \qquad S^i(\hat{Z}^i, \lambda) = \max_{Z^i \in A^i} S^i(Z^i, \lambda)$$

Condition (4) will be termed the condition for matched planning and the plans that meet these conditions will be referred to as matched plans. Note that the matched plans are generally not mutually profitable. Suppose that the procedure to represent the preference function has been set in the system (e.g. the function can be defined by setting a finite number of parameters). Thus a certain class H^i of feasible preference function has been identified. This class will not necessarily contain the objective function $\eta^i(Z^i, \lambda)$. Assume first that $\eta^i(Z^i, \lambda) \in H^i$, $i = 1, 2, ..., m$.

Denote as $A^i(\lambda)$ a set of mutually profitable plans and as $A^i_S(\lambda)$ a set of matched plans under the preference function $S^i(Z^i, \lambda) \in H^i$, $\lambda \in L$. We shall further use the assumption of the slight effect of the information, reported by each AE, on the future control $\lambda \in L$. This assumption will be written in the following form: for any $S^i(Z^i, \lambda) \in H^i$, $Z^i \in A^i_S(\lambda)$ and $\lambda \in L$, there are $S^j(Z^j, \lambda) \in H^j$, $j \neq i$, such that in the optimal solution to the problem of matched planning, λ is the control and the plan for the i-th AE is Z^i. The point of the above assumption is that an AE, no matter what information it reports can expect any control λ from L and any matched plan from $A^i_S(\lambda)$.

Thence it follows that the necessary condition for the mutual profitability of matched plan is $A^i_S(\lambda) \subseteq A^i(\lambda)$, (since $A^i \neq \emptyset$, then $A^i_S(\lambda) \neq \emptyset$ at any $\lambda \in L$). A simple sufficient condition for mutual profitability of a matched plan is the equality between the preference function and the objective function

(with the accuracy to a positive multiplier). For an optimal matched plan to be an optimal mutually profitable plan it is sufficient that from the condition $A^i_S(\lambda) \subseteq A^i(\lambda)$ follow $A^i_S(\lambda) = A_i(\lambda)$. (This requirement is a necessary one in that if $A^i_S(\lambda) \subset A^i(\lambda)$, then there is a problem of matched planning where optimal matched plan is not an optimal mutually profitable plan.)

Theorem 1. *In order that for any $\eta^i(Z^i, \lambda) \in H^i$ from the condition $A^i_S(\lambda) \subseteq A^i(\lambda)$ follow $A^i_S(\lambda) = A^i(\lambda)$ it is necessary and sufficient that for any $S^i_1(Z^i_1, \lambda)$ and $S^i_2(Z^i, \lambda)$ from H^i, the relation $A^i_{S_1}(\lambda) \not\subset A^i_{S_2}(\lambda)$ would take place.*

Necessity. Let $A^i_{S_1}(\lambda) \subset A_{S_2}(\lambda)$ and the objective function $\eta^i(Z^i, \lambda) = S^i_2(Z^i, \lambda)$. Then $A^i(\lambda) = A^i_{S_2}(\lambda)$ and $A^i_{S_1}(\lambda) \subset A^i(\lambda)$.

Sufficiency. Let $A^i_{S_1}(\lambda) \not\subset A^i_{S_2}(\lambda)$ for any $S^i_1(Z^i, \lambda)$ and $S^i_2(Z^i, \lambda)$ from H^i. Then from $\eta^i(Z^i, \lambda) \in H^i$ it also follows that $A^i_S(\lambda) \not\subset A^i(\lambda)$ at any $S^i(Z^i, \lambda) \in$ $\in H^i$.

Theorem 1 imposes certain constraints on the selection of the preference function form (i.e. on the class H^i). Therefore one should specially test the specific classes H^i on their consistency with the conditions of Theorem 1 and justify the application of the "fair play" principle to solve the problem of mutually profitable planning.

Consider necessary justification in the case to which the examples that follow can be reduced. Assume that

$$A^i: \sum_{j=1}^{n} Z^i_j = T_i\,, \qquad T_i > 0\,, \quad i = 1, 2, ..., m\,;$$

$$\eta^i(Z^i, \lambda) = \sum_{j=1}^{n} S^i_j \lambda_j Z^i_j\,; \quad S^i_j \geqslant 0\,, \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., n\,.$$

$$L: \sum_{j=1}^{n} \lambda_j = 1\,, \quad \lambda_j \geqslant 0\,, \quad j = 1, 2, ..., n\,.$$

Assume that the preference function

$$S^i(Z^i, \lambda) = \sum_{j=1}^{n} \lambda_j \tau^i_j Z^i_j\,.$$

Then each AE reports a preference vector τ^i such that $\tau^i_j \geqslant 0, i = 1, 2, ..., m$;

$j = 1, 2, ..., n$. Without limiting the generality we can assume that

$$\sum_{j=1}^{n} S_j^i = \sum_{j=1}^{n} \tau_j^i = 1\,, \quad i = 1, 2, ..., m\,. \tag{5}$$

Thus, the class H^i is defined by a set of τ_j^i, satisfying (5).

Theorem 2. *To make a matched plan mutually profitable it is necessary and sufficient to have* $\tau_j^i = S_j^i$, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$.

Proof. The sufficiency of the condition is evident. Let us prove the necessity. Assume $\tau_{j1}^i > S_{j1}^i$ for some j_1. It follows that there is a j_2 for which $\tau_{j2}^i < S_{j2}^i$. Consider a control λ which satisfies the conditions

$$\lambda_j \cdot \tau_j^i < \max(\lambda_{j1}, \tau_{j1}^i, \lambda_{j2}, \tau_2^i), \quad j \neq j_1, j_2\,.$$

$$\frac{\tau_{j2}^i}{\tau_{j1}^i} < \frac{\lambda_{j1}}{\lambda_{j2}} < \frac{S_{j2}^i}{S_{j1}^i}\,.$$

Then in a matched plan $Z_{j1}^i = T_i$, $Z_j^i = 0$, $j \neq j_1$, while in a mutually profitable plan $Z_j^i = 0$, $j \neq j_2$, $Z_{j2}^i = T_i$.

Thus in this case the necessary and sufficient condition for the mutual profitability of a plan is the preference function equal to the objective function.

Note. The assumption that the objective function belongs to the class H^i will not generally be true because the objective function of an AE may be complicated enough and be dependent on quite a great number of parameters, human and social factors included. However, since it can select the preferences from the class H^i alone, an AE has to adapt to the conditions of the system functioning. Therefore an AE selects from the class H^i a preference function which best represents its actual interests and use it as the objective function under the given conditions.

This assumption is of course realistic if the class H^i does include a preference function which represents the aims of the AE sufficiently. Otherwise contradictions may occur in the system and as a result the system operation can become unstable. In that case the class H^i should be changed.

4. Probabilistic models

Let the execution of each plan Z^i be specified by the values of the parameter vector x^i which is a random quantity with the distribution $F^i(x^i)$. In this case the set A^i will be represented by a set of plans which correspond to certain estimates of random parameters. These are generally either the mean values of parameters or the values with a given probability to exceed them. Assume that an AE is aware of the required estimates of random parameters while the CU is not or knows them just approximately. Denote by a^i the vector of random parameter estimates reported to the CU by the i-th AE. To verify the truthfulness of the reported estimates certain assumptions on the AE objective functions should be made to assume that the objective functions $\varphi^i(x^i, a^i, Z^i, \lambda)$ depend in a known way on the parameter vector x^i under the observation, vector of estimates a^i, the plan Z^i and the control λ. This case represents the control problem in economic systems where $\varphi^i(x^i, a^i, Z^i, \lambda)$ determines, for instance, the profit. At the information reporting stage the mean value of φ^i is taken as the objective function of an AE

$$M^i(a^i, Z^i, \lambda) = \int \dots \int \varphi^i(x^i, a^i, Z^i, \lambda)\, dF^i(x^i) .$$

If the maximal guaranteed value of M^i is used as an objective function the estimates a^i of the parameters reported by an AE to the CU are found from the condition of the maximum

$$\xi^i(a^i) = \min_{\lambda \in L} \max_{Z^i \in A^i} M^i(a^i, Z^i, \lambda) .$$

Let us take up two specific cases that are important to further applications.

(6) a) $$\varphi^i(x_1^i, a^i, Z^i, \lambda) = \sum_{j=1}^{n} \lambda_j \, [x_j^i - \alpha_j^i (x_j^i - a_j^i)^2] \, Z_j^i ,$$

where $\alpha_j^i > 0$ ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$) are constant numbers. By maximizing the mean value of φ^i over a_j^i we easily obtain

$$a_j^i = \int \dots \int x_j^i \, dF^i(x^i) , \quad j = 1, 2, \dots, n ,$$

which means that each AE reports the mean values of the parameters to the CU. Then

$$M^i(a^i, z^i, \lambda) = \sum_j \lambda_j (a^i_j - \alpha^i_j D^i_j) Z^i_j ,$$

where

$$D^i_j = \int \ldots \int (x^i_j - a^i_j)^2 \, dF(x^i) .$$

Denote $S^i_j = a^i_j - \alpha^i_j D^i_j$. Then by Theorem 2 the preference function reported by the AE to the CU is equal (with the accuracy to the positive multiplier) to

$$S^i(Z^i, \lambda) = \sum_{j=1}^{n} \lambda_j S^i_j Z^i_j . \tag{7}$$

b) $$\varphi^i(x^i, a^i, Z^i, \lambda) = \sum_{j=1}^{n} \lambda_j [x^i_j - \alpha^i_j (a^i_j - x^i_j)] Z^i_j , \quad \text{if } x^i_j \leqslant a^i_j ,$$

$$= \sum_{j=1}^{n} \lambda_j [x^i_j - \beta^i_j (x^i_j - a^i_j)] Z^i_j , \quad \text{if } x^i_j \geqslant a^i_j , \tag{8}$$

where α^i_j, β^i_j are positive numbers.

Maximizing the mean value of φ^i over a^i_j we find that the estimate a^i_j reported to the CU satisfies the equation:

$$F^i_j(a^i_j) = \frac{\beta^i_j}{\alpha^i_j + \beta^i_j} ; \quad i = 1, 2, \ldots, m ; \quad j = 1, 2, \ldots, n . \tag{9}$$

where $F^i_j(x^i_j)$ is the x^i_j distribution function. The preference function reported to the CU will also have the form of eq. (7) where

$$S^i_j = \int_{-\infty}^{a^i_j} [x^i_j - \alpha^i_j (a^i_j - x^i_j)] \, dF^i_j(x^i_j) + \int_{a^i_j}^{\infty} [x^i_j - \beta^i_j (x^i_j - a^i_j)] \, dF^i_j(x^i_j) .$$

These two kinds of the objective functions of AE will be considered in further application examples. In the models discussed below the problem of matched planning is considered. By Theorem 2, however, a solution to this problem determines the optimal mutually profitable plan.

5. Models of Matched planning

Consider some simple models of economic systems. The AE's will be either elements of a system that manufacture (consume) certain types of products or elements capable of performing certain jobs. The models of AE's are presumed given with the accuracy to a finite number of unknown parameters. Plans are developed for a specified period. In different models the unknown parameters are the productivities of elements in terms of either different products or different kinds of job, the amount of crops to be received from a unit area, etc. The profit of an AE is assumed to be determined by eqs. (6) or (8) and therefore each AE reports to the CU the appropriate estimates of the parameters a_j^i and the coefficients S_j^i of the preference function.

5.1. *Production planning*

Let us consider a system which consists of m manufacturers that are capable of making n types of products.

Denote by T_i the operational time of the i-th manufacturer over the period covered by the plan. Let a_j^i correspond to the estimate of the productivity of the ith AE by the jth kind of product, Z_j^i is the operational time spent by the ith manufacturer to make the jth type of product. Denote by C_j the value of the jth product unit (e.g. its price at the world marked) by b_{jk}^i the amount of the k-th product type required to produce a unit of the j-th product type by the i-th manufacturer. To simplify the model assume that any amount of the product of any type can be purchased by the unit price C_j ($j = 1, 2, ..., n$). The problem is to find a matched plan Z and control λ that ensure the maximal profit for the system. The profit can be expressed as:

$$\Phi(Z, \lambda) = \sum_{j=1}^{n} \sum_{i=1}^{m} Z_j^i a_j^i (C_j - \sum_{k=1}^{n} C_k b_{jk}^i) = \sum_{i=1}^{m} \sum_{j=1}^{n} C_j^i Z_j^i \ ,$$

where

$$C_j^i = a_j^i (C_j - \sum_{k=1}^{n} C_k \cdot b_{jk}^i)$$

under the constraints

$$\sum_{j=1}^{n} Z_j^i \leqslant T_i \ , \quad i = 1, 2, ..., m \ , \tag{10}$$

$$(\max_k \lambda_k S_k^i - \lambda_j \cdot S_j^i) Z_j^i = 0, \quad i = 1, 2, ..., m, \ j = 1, 2, ..., n \ .$$

Note that the optimal solution to the problem without the matching requirement is determined very easily: let $C^i_{j1} = \max_j C^i_j$, then

$$Z^i_j = \begin{cases} T_i\,, & \text{if } j = j_1\,, \\ 0\,, & \text{if } j \neq j_1\,; \end{cases}$$

(if there are several maximal C^i_j any one is taken).

Denote by Q_j a set of AE's manufacturing the j-th type of product under the optimal plan. To make the problem more specific assume that there is a single optimal plan. Calculate

$$\epsilon_{kj} = \max_{i \in Q_j} \frac{S^i_k}{S^i_j} \quad (\text{if } Q_j = \emptyset \text{ then } \epsilon_{kj} = 0)\,,$$

and define a complete graph G where the arc gain is equal to ϵ_{kj}. The cycle gain then is a product of gains of arches which make the cycle.

Theorem 3. *For complete matching it is necessary and sufficient to have no cycles with the gain in excess of 1 in the graph.*

Proof. The matching conditions for an optimal plan have the form

$$\lambda_j \geqslant \lambda_k \epsilon_{kj}\,, \quad \epsilon_{jj} = 1\,, \quad k, j = 1, 2, \ldots, n.$$

Denote

$$\mu_j = \ln \lambda_j\,, \qquad \xi_{kj} = \ln \epsilon_{kj}\,.$$

Then the conditions transform to

$$\mu_j \geqslant \mu_k + \xi_{kj}\,.$$

These are the conditions closely reminding a problem of the graph vertices potentials when the arch lengths are equal to ξ_{kj}.

This problem is known to be solvable if there are no cycles of positive length in the graph. Consequently, no cycles with gain above 1 should be present in the initial graph.

Example. Let the matrices C_j^i and S_j^i have the form

$$(C_j^i) = \begin{array}{c|ccc} j \backslash i & 1 & 2 & 3 \\ \hline 1 & 2 & 3 & 1 \\ 2 & 5 & 0 & 3 \\ 3 & 1 & 6 & 4 \\ 4 & 0 & 1 & 5 \end{array} \qquad (S_j^i) = \begin{array}{c|ccc} j \backslash i & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 5 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & 1 & 2 \\ 4 & 0 & 6 & 1 \end{array}$$

The optimal solution without the matching requirement is $Z_2^1 = Z_3^2 = Z_4^3 = 1$, otherwise $Z_j^i = 0$, $\Phi_0 = 16$. Calculate

$$\epsilon_{32} = \frac{S_3^1}{S_2^1} = 2\,; \quad \epsilon_{23} = \frac{S_2^2}{S_3^2} = 0\,; \quad \epsilon_{24} = \frac{S_2^3}{S_4^3} = 6\,;$$

$$\epsilon_{42} = \frac{S_4^1}{S_2^1} = 0\,; \quad \epsilon_{43} = \frac{S_4^2}{S_3^2} = 6\,; \quad \epsilon_{34} = \frac{S_3^3}{S_4^3} = 2\,.$$

The cycle (3, 2, 4, 3) has the gain equal to 72. The condition of complete matching is not thus met.

The optimal matched solution is: $Z_3^2 = 1$, $Z_2^3 = 1, Z_2^1 = 1$, otherwise $Z_j^i = 0$, $\Phi_b = 14$. The matching coefficient is $\rho = \frac{7}{8}$.

5.2. *Distribution of arable areas*

Let a system consist of m state farms. Denote by T_i the arable area of the i-th farm, by a_j^i the estimate of the productivity of the j-th type of the crop in the i-th farm, by A_j the relative need in the j-th kind of crops by Z_j^i the area alloted for the j-th crop by the ith farm.

The problem is to find λ and Z that would maximize the production yield under the given proportions of product types. The minimal relation of the amount produced to the relative need will be denoted as β. The problem is formulated as follows:

Maximize β at the constraints

$$\sum_{j=1}^{n} Z_j^i \leqslant T_i\,; \quad i = 1, 2, ..., m\,; \quad \sum_{i=1}^{m} a_j^i Z_j^i \geqslant \beta A_j\,; \quad j = 1, 2, ..., n\,;$$

$$(\max_k \lambda_k S_k^i - \lambda_j S_j^i)\, Z_j^i = 0\,, \; i = 1, 2, ..., m, \; j = 1, 2, ..., n.$$

Note that the matching conditions in this case coincide with the conditions used in the preceding problem. If in the optimal solution without the matching requirement to denote the set of farms that intend to seed the crop j as Q_j ($Q_j = \{i: Z_j^i > 0\}$) the conditions of Theorem 3 will be the necessary and sufficient conditions of complete matching for the problem at hand. The simple sufficiency conditions for complete matching is the equivalence (with the accuracy to the positive multiplier) of the estimates a_j^i and the preferences S_j^i [8].

5.3. *The problem of supply*

Let us consider a system which consists of m suppliers and q customers. The customer k orders B^k units of products and informs on his preferences τ_j^k with regard to each type of product. The types of products for which $\tau_j^k > 0$ are assumed interchangeable for the customer if supplied in equal amounts, in other words, the total amount of different kinds of products is B^k. Denote as we did before by a_j^i the estimate of the i-th supplier productivity in terms of the j-th kind of product, by S_j^i his preferences, by Z_j^i operational time when engaged in the manufacture of the j-th kind of product. Further U_j^k will denote the amount of product of the j-th kind delivered to the customer k, $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ the control vector for the suppliers, $\pi = (\pi_1, \pi_2, ..., \pi_q)$ the same for the customers. Finally, C_j will denote the unit price of the j-th product at the market outside the system.

The problem is stated as follows: determine λ, π, Z, U that minimize

$$C = \sum_{j=1}^{n} C_j \Delta_j ,$$

at the constraints

$$\sum_{j=1}^{n} U_j^k = B^k , \quad k = 1, 2, ..., q . \tag{11}$$

$$\sum_{i=1}^{m} a_j^i Z_j^i + \Delta_j \geqslant \sum_{k=1}^{q} U_j^k ; \quad \sum_{j=1}^{n} Z_j^i \leqslant T_i , \quad i = 1, 2, ..., m . \tag{12}$$

$$(\max_p \lambda_p \cdot S_p^i - \lambda_j \cdot S_j^i) Z_j^i = 0 . \tag{13}$$

$$(\max_p \pi_p \tau_p^k - \pi_j \tau_j^k) U_j^k = 0 . \tag{14}$$

where $i = 1, 2, ..., m$; $j = 1, 2, ..., n$; $k = 1, 2, ..., q$. If in the optimal solution $C = 0$, the demand can be met by the system's own potential alone. In this case Δ_j will denote the surplus of the j-th kind of products and C_j the profit gained by marketing a unit of the j-th product. The problem is then to maximize

$$C = \sum_{j=1}^{n} C_j \Delta_j$$

at constraints (11), (13), (14) and

$$\Delta_j = \sum_{i=1}^{m} a_j^i Z_j^i - \sum_{k=1}^{q} U_j^k \geqslant 0 .$$

Note that the necessary and sufficient conditions for complete matching are also determined by Theorem 3 and should be met for both the preferences of customers and suppliers.

The simple sufficiency conditions are:

$$\tau_j^k > 0$$

$$a_j^i = \delta \cdot S_j^i \qquad (\delta \text{ is a positive multiplier})$$

for all $i = 1, 2, ..., m$; $j = 1, 2, ..., n$; $k = 1, 2, ..., q$.

6. Conclusions

The concept of active systems permits to represent the salient features of man-machine systems such as their "own" objective functions, reporting information on potential available turning objective functions to account, the use of information on controlling subsystems, etc.

The basic difficulty in solving the control problem has been the insufficient information on the potential and intentions of controlled subsystems. The principle of fair play surmounts this difficulty to a certain extent. The problems arising then such as the selection of the classes H^i and the effect of this selection on the functioning of an AS. Application of the fair play principle to various models of AS's optimal synthesis of multi-level AS's opens an new challenging fields for research.

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