Control
and Cybernetics

VOL. 1 (1972) No. 1/2

Problems of optimum distribution of resources

by

V. N. BURKOV

Institute of Control Problems,
Moscow

The optimum distribution of a limited quantity of resources, is one of the most important trend in the theory of network planning and of control. Problems of an optimum distribution of resources, are in principle extremal problems of combinational type. At present there are no effective and accurate methods for the solution of such problems. A satisfactory developed theory exists only for the problems where ordering of the network events is assumed. The paper considers basic results and methods of optimum distribution of resources, when the network events are ordered.

1. Basic notions

Operation-process described by an equation of the planning type

$$h(t)=\frac{dx(t)}{dt}=f[\bar{u}(t)], \tag{1}$$

where: $h(t)$ — operation rate at a moment t; $x(t)$ — operation state at a moment t; $\bar{u}(t)=(u_1(t), u_2(t), ..., u_m(t))$ — distribution of resources for an operation, at a moment t (m — number of kinds of resources); $f(\bar{u})$ — dependence of operation rate upon the quantity of resources (rightside continuous, no decreasing function of $\bar{u}$, with $f(\bar{0})=0$).

The operation is completed at the moment t, if $x(t)=w>0$. W is called the operation volume. Let's assume an initial moment $t=0$. The moment of completion is determined in accordance with (1) as the minimum t which satisfies the equation

$$\int_0^t f[\bar{u}(\tau)]\, d\tau=w. \tag{2}$$

Usually it is assumed that resources of different kinds are taking part in the operation with given proportions, i.e. $u_j(t)=\alpha_j v(t)$, where $\alpha_j \geqslant 0$ — given numbers. The distribution $\bar{u}(t)=\alpha v(t)$ is called the allocation of resources; $\bar{\alpha}$ — is called the vector of the allocation parameters; $v(t)$ — the intensity of allocation at the moment t.

The notion "distribution of resources" simplifies the description of the operation. Actually, for a given $\bar{\alpha}$ the operation rate depends only on the intensity, i.e. $h(t) = f[v(t)]$. A finite, partly ordered set of operations, forms an operation complex. Each operation from the complex is determined by its volume w_i, by the dependence of the operation rate $f_i(v_i)$ upon the intensity v_i, and by the allocation parameters $\{\alpha_{ij}\}$, $j=1, 2, ..., n$, $i=1, 2, ..., n$.

Furthermore advantage is taken from a method according to which the complex is considered as a network where the arcs correspond to operations, the nodes to the events, and the dotted arcs to the relations.

The network events are said to be ordered, when their occurrence sequence is given. Assuming that the consecutive numbers of the events, correspond to the sequence of their occurrence, the required condition of ordering of events can be written in the following form $t_0 \leqslant t_1 \leqslant t_2 \leqslant ... \leqslant t_r$ where t_r is the moment of the occurrence of the r^{th} event, $(r+1)$-quantity of network events.

2. Formulation of the problem

Let's denote by $N_j(t)$ the quantity of resources of the j^{th} kind at the moment t, $j=1, 2, ..., m$.

Problem 1. Determine $v_i(t) \geqslant 0$, $i=1, 2, ..., n$, which satisfies the constraints

$$\sum_{i=1}^{n} \alpha_{ij} v_i(t) \leqslant N_j, \quad j=1, 2, ..., m, \tag{3}$$

in such a way, that the complex be carried out in the minimum time T.

Effective and accurate methods for solution of problem 1 are known, but only for special cases. The network events are assumed to be ordered.

3. Independent operations

The operations of a complex are independent when they can be carried out simultaneously (in the presence of satisfactory quantity of resources). The case of independent operations apart from being interesting, is very important for the consideration of complexes in a general form. Let's assume initially, that $f_i(v_i)$ are the convex (to top) functions of v_i $(i=1, 2, ..., n)$.

Theorem 1. There exists an optimum solution in which $v_i(t) = \text{const.}$, $t \in [0, T]$, $i=1, 2, ..., n$, i.e. all the operations are worked out with the same intensity. The idea of the theorem proof can be seen from the following:

Let $\bar{v}(t) = (v_1(t), ..., v_n(t))$ be the optimum solution. We will determine the distribution of resources

$$\bar{v}_i^* = \varphi_i\left[\frac{1}{T}\int_0^T f_i[v_i(\tau)]\,d\tau\right] = \varphi_i(w_i/T), \quad i=1, 2, ..., n,$$

where φ_i is the function reciprocal to f_i. Since f_i is convex (to top), φ_i is convex (to bottom). Therefore

$$v_i^* = \varphi_i\left[\frac{1}{T}\int_0^T f_i[v_i(\tau)]\,d\tau\right] \leqslant \frac{1}{T}\int_0^T v_i(\tau)\,d\tau.$$

Let's note that for the distribution $\bar{v}^*$ all the operations will be worked out. Moreover, since

$$\sum_{i=1}^{n} \alpha_{ij} v_i^* \leqslant \frac{1}{T}\int_0^T \left[\sum_{i=1}^{n} \alpha_{ij} v_i(\tau)\right] d\tau \leqslant N_j, \quad j=1, 2, \ldots, m,$$

$\bar{v}^*$ will be the admissible distribution of resources. This proves the theorem. According to theorem 1 the minimum duration time of the complex $T_{\min}$, is determined as the minimum T which satisfies the set of inequalities:

$$\sum_{i=1}^{n} \alpha_{ij}\, \varphi_i(w_i/T) \leqslant N_j, \quad j=1, 2, \ldots, m. \tag{4}$$

Example 1. Let $h_i = f_i(v_i) = v_i^{1/\alpha}$, $\alpha > 1$, $i=1, 2, \ldots, n$. We have

$$\sum_{i=1}^{n} \alpha_{ij}(w_i/T)^{\alpha} \leqslant N_j, \quad j=1, 2, \ldots, m,$$

$$T_{\min} = \max_j \left(\frac{1}{N_j}\sum_{i=1}^{n} \alpha_{ij}\, w_i^{\alpha}\right)^{1/\alpha}.$$

And let $f_i(v_i)$ be arbitrary functions.

Theorem 2 [1]. There is an optimum solution in which $\bar{v}(t)$ — is a piecewise constant function with no more than n intervals.

Proof. A conversion of variables $\bar{h} = f(\bar{v})$ will be carried out. Let's denote by H the set of admissible $\bar{h}$. This set is determined by the inequality

$$\sum_{i=1}^{n} \alpha_{ij}\, \varphi_i(h_i) \leqslant N_j, \quad j=1, 2, \ldots, m,$$

and is no more convex.

Let $\hat{H}$ be the convex closure of H. The optimum solution of the problem will be determined when regarding $\hat{H}$ as the admissible set of $\bar{h}$. In accordance with theorem 1 the solution denoted as $\bar{h}^0$, is a constant value vector function on $(0, T)$. $\bar{h}^0$ will be represented in the form of a linear convex combination of no more than n points $\bar{h}^1, \bar{h}^2, \ldots, \bar{h}^n$ from the set H. The resulting set of resources distributions $v_i^j = \varphi_i(h_i^j)$, $i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$, determines the optimum solution of the problem.

Let's denote by $T_{\min}(\bar{w})$ the minimum duration time of the complex as a function of operation volume $\bar{w} = (w_1, w_2, \ldots, w_n)$.

Theorem 3. T_{min} is a convex (to bottom) function of its own arguments.

Proof. The set H can be considered as convex (see theorem 2) without limiting its generality. Let

$$\bar{h}^1 = \frac{\bar{w}^1}{T_{min}(\bar{w}^1)}, \quad \bar{h}^2 = \frac{\bar{w}^2}{T_{min}(\bar{w}^2)},$$

$$\bar{w} = \alpha \bar{w}^1 + (1-\alpha)\,\bar{w}^2, \quad 0 \leqslant \alpha \leqslant 1.$$

The time of complex duration will be determined for

$$\bar{h} = \beta \bar{h}^1 + (1-\beta)\,\bar{h}^2 = \frac{\bar{w}}{T} = \frac{1}{T}\,[\alpha \bar{h}^1\,T_{min}(\bar{w}^1) + (1-\alpha)\,\bar{h}^2\,T_{min}(\bar{w}^2)].$$

When equating the coefficients of $\bar{h}^1$ and $\bar{h}^2$ we will obtain:

$$T\beta = \alpha T_{min}(\bar{w}^1), \quad T(1-\beta) = (1-\alpha)\,T_{min}(\bar{w}^2),$$

and finally

$$T_{min}(\bar{w}) \leqslant T = \alpha T_{min}(\bar{w}^1) + (1-\alpha)\,T_{min}(\bar{w}^2).$$

The theorem is prooved.

4. Networks with ordered events

Let's denote by R_s — the set of operations which can be worked out during the s^{th} interval (interval between the $(s-1)^{th}$ and the s^{th} event, $s=1, 2, \ldots, r$); Q_i the set of intervals during which the i^{th} operation can be worked out; $\Delta_s(z_s)$, where $z_s = \{x_{is}: i \in R_s\}$, the minimum length of the s^{th} interval as a function x_{is}, $(i \in R_s)$; $x_{is} \geqslant 0$ the volume of the i^{th} operation worked out during the s^{th} interval. The time of complex duration will be

$$T(z) = \sum_{s=1}^{r} \Delta_s(z_s).$$

Theorem 4. $T(z)$ is a convex (to bottom) function of x_{is} $(i \in R_s,\ s=1, 2, \ldots, r)$.

The proof results from theorem 3, because the operations from the set are independent.

Problem 2. Determine the non-negative $\{x_{is}\}$, $i \in R_s$, $s=1, 2, \ldots, r$, which satisfy conditions

$$\sum_{s \in Q_i} x_{is} = w_i, \quad i=1, 2, \ldots, n, \tag{5}$$

and minimize $T(z)$.

Let's note that theorems 2 and 3 hold good also for the case of ordered events. This is obvious for theorem 2. A theorem similar to theorem 3 will be proved.

Theorem 5. If the network events are ordered, $T_{min}(\bar{w})$ is a convex (to bottom) function of w_i, $i=1, 2, \ldots, n$.

Proof. Let $z^1=\{x_{is}^1\}$ be the optimum solution of the problem for a vector of operations volumes $\bar{w}^1$, and $z^2=\{x_{is}^2\}$ the optimum solution for a vector of operations volume $\bar{w}^2$. Let $\bar{w}=\alpha\bar{w}^1+(1-\alpha)\bar{w}^2$, $0\leqslant\alpha\leqslant 1$. In such a case $z=\{x_{is}\}=\{\alpha x_{is}^1+(1-\alpha)x_{is}^2\}$, $i\in R_s$, $s=1,2,\dots,r$, are admissible because,

$$\sum_{s\in Q_i} x_{is}=\alpha w_i^1+(1-\alpha)w_i^2=w_i, \qquad i=1,2,\dots,n.$$

We have

$$T(z)=T\big(\alpha z^1+(1-\alpha)z^2\big)\leqslant\alpha T(z^1)+(1-\alpha)T(z^2)=\alpha T_{\min}(\bar{w}^1)+(1-\alpha)T_{\min}(\bar{w}^2).$$

Finally we obtain

$$T_{\min}(\bar{w})\leqslant T(z)\leqslant\alpha T_{\min}(\bar{w}^1)+(1-\alpha)T_{\min}(\bar{w}^2).$$

The convexity of function $T(z)$ and the linearity of constraints (5) make it possible to obtain the necessary and sufficient conditions of optimality for different problems.

Example 2. Let all the operations be worked out by means of resources of one kind, in a quantity N. It follows from (4) that Δ_s is determined from the equation

$$\sum_{i\in R_s}\varphi_i(x_{is}/\Delta_s)=N. \tag{6}$$

The necessary and sufficient conditions of optimality will be obtained by differentiating (6) with respect to x_{is}

$$\frac{1}{\lambda_i}\frac{d\varphi_i(h_{is})}{dh_{is}}=\sum_{j\in R_s}h_{js}\frac{d\varphi_j(h_{js})}{dh_{js}}, \tag{7}$$

where $h_{is}=x_{is}/\Delta_s$. We will obtain the necessary conditions for an existence of optimum solution, when the rate of performing each operation is constant, i.e. $h_{is}=h_i$, $s\in Q_i$, $i=1,2,\dots,n$, for an arbitrary complex of operations. Let the network be such, that all $\Delta_s>0$, $s=1,2,\dots,r$. Then, the quantity of resources $\{v_i\}$ represent a stream of value N in the network. It follows from (7) that functions $F_i(v_i)=\frac{h_i\,d\varphi_i(h_i)}{dh_i}=f_i(v_i)\left[\frac{df_i(v_i)}{dv_i}\right]^{-1}$ should also represent a stream for a network of any type, and for any volume of operations. In (4) it was shown that $\{F_i(v_i)\}$ is the stream for whichever stream $\{v_i\}$ if and only if $F_i(v_i)=a_i+\alpha v_i$, where $\{a_i\}$ is the stream.

Solving the equation

$$\frac{1}{h_i}\frac{dh_i}{dv_i}=\frac{1}{a_i+\alpha v_i},$$

we obtain

$$h_i=c_i(a_i+\alpha v_i)^{1/\alpha}, \qquad c_i>0,\ i=1,2,\dots,n.$$

Since $f_i(0)=0$, then $a_i=0$, $i=1,2,\dots,n$.

It follows from the conditions of convexity $f_i(v_i)$ that $\alpha\geqslant 1$. Finally $f_i(v_i)=c_i v_i^{1/\alpha}$ is obtained. We can always adopt a coefficien c_i equal 1, changing adequately the operation volume. By means of a direct substitution to (7) it can be shown that conditions

$$f_i(v_i)=v_i^{1/\alpha},\ \alpha>1, \qquad i=1,2,\ldots,n$$

are the necessary and sufficient conditions for an optimum solution existence, when the rate of performing each operation in the optimum solution is constant for arbitrary operations volumes, and an arbitrary network.

The case of power relations is in a certain way unique. If all the operations are worked out by means of resources of one kind, it is possible to determine a certain value w_e, called the equivalent volume of the complex, so that $T_{\min}$ is determined from the equation:

$$\int_0^{T_{\min}} N^{1/\alpha}(\tau)\,\mathrm{d}\tau = w_e.$$

Besides, if the time of operation duration τ_i depends on the losses s_i, suffered from its performance in the following way:

$$s_i=\frac{w_i^\alpha}{\tau_i^{\alpha-1}}, \qquad i=1,2,\ldots,n,\ \alpha>1,$$

then the minimum time of complex duration for a limited value of losses $\sum_{i=1}^{n} s_i \leqslant S$, is determined by the equation

$$T_{\min}=(w_e^\alpha/s)^{1/(\alpha-1)}.$$

The equivalent volume of the complex depends only on the operations volumes and the network structure. For a network composed by operations ordered in series $w_e=\sum_{i=1}^{n} w_i$. For a network composed of operations ordered in parallel $w_e=\left(\sum_{i=1}^{n} w_i^\alpha\right)^{1/\alpha}$.

5. Fixed intensities

It is said that an operation is performed with a fixed intensity, when the quantity of resources for that operation can have only one value. Let's denote by τ_i the time of the i^{th} operation duration; β_{ij} the fixed quantity of resources of the j^{th} kind in the j^{th} operation. We can attribute to an arbitrary set of independent operations a vector $\bar{x}=(x_1, x_2, \ldots, x_n)$, where $x_i=1$, if the i^{th} operation belongs to this set. In the contrary case $x_i=0$. Each vector, which satisfies the condition

$$\sum_{i=1}^{n} x_i\,\beta_{ij}\leqslant N_j, \qquad j=1,2,\ldots,m, \tag{8}$$

will be called an admissible vector. Let $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^q$ be a set of admissible vectors. We will denote $u_s\geqslant 0$ the time of duration of the interval during which the operations corresponding to vectors $\bar{x}^s$ are performed.

Problem 3. Determine u_s, $s=1, 2, \ldots, q$, which satisfies the condition

$$\sum_{s=1}^{q} \bar{x}_i^s u_s = \tau_i, \qquad i=1, 2, \ldots, n, \tag{9}$$

and minimizes $T=\sum_{i=1}^{q} u_s$.

It is a problem of linear programming where the matrix of conditions is given in an implicit form (8). Let $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$, be the basic vectors of a certain preliminary solution. We will denote by $\bar{y}^i$ a vector $(x_1, x_2, \ldots, x_n)$, so that $x_i=1$, $x_j=0$, $j \neq 1$. We will express $\bar{y}^i$ by means of the basic vectors $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$. Let

$$\bar{y}^i = \sum_{j=1}^{n} c_{ij} \bar{x}^i \tag{10}$$

and denote $a_i = \sum_{j=1}^{n} c_{ij}$.

Problem 4. Determine $x_i=0, 1$ which satisfies (8) and maximizes

$$c = \sum_{i=1}^{n} a_i x_i. \tag{11}$$

If in the optimum solution of problem 4, $c \leqslant 1$, the solution $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$ is optimum too. In the contrary case the optimum solution of problem 4 is determined by the vector which must be introduced to the base according to the simplex method.

Note 1. As the admissible vector determines the set of independent operations, the problems (8), (11) can be solved for each set R_s individually.

Note 2. To prevent looping, the vectors eliminated from the base must be kept in mind if $u=0$ corresponds to them as long as a vector with $u>0$ will not be eliminated.

The problem (8), (11) is a problem of linear integer programming with variables 0, 1, and in the general case there are no effective methods for its solution. Let us consider the case when problem (8), (11) takes a simple form.

Let all the operations of the complex be divided into classes in such a way that operations of the j^{th} class are worked out only by means of the j^{th} kind of resources in a quantity β_i $(i=1, 2, \ldots, n)$.

We will denote by P_{js} the set of operations of the j^{th} class, which can be worked out in the s^{th} interval. In this case the problem (8), (11) divides into separate problems for each $P_{js} \neq 0$, of the following form:

$$c_{js} = \sum_{i \in P_{js}} a_i x_i \to \max, \qquad \sum_{i \in P_{js}} \beta_i x_i \leqslant N_j. \tag{12}$$

The problem (12) is known as the problem of packing, and can be solved effectively enough by means of the method of dimensions and constraints or the method of dynamic programming.

The preliminary solution $\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n$, is the optimum one if

$$c=\max_s \sum_{j=1}^{m} c_{js} \leqslant 1, \quad c_{js}=0 \quad \text{if} \quad P_{js}=\varnothing .$$

Problem (12) can be solved in an elementary way if all $\beta_i=1$. Obviously, in this case c_{js} is equal to the sum N_j of the positive maximum values of a_i (or simply, of all positive a_i, if their quantity is smaller than N_j).

Example 3. The data for the set shown in Fig. 1 (the operations consecutive numbers are noted on the arcs) are given below. All the operations are worked out by means of $N=11$ resources of one kind:

i	1	2	3	4	5
τ_i	2	4	6	5	2.
β_i	6	5	4	4	7

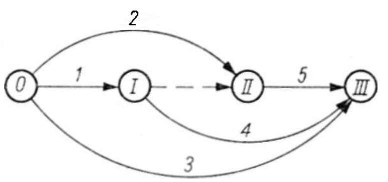

Fig. 1

Obtaining of a preliminary solution. To obtain a preliminary solution, we will apply a heuristic algorithm according to which, first start the operations which have the smallest values of the latest possible time of starting t_i^n.

In the case of similar t_i^n we can start any operation. The values of t_i^n are given below

i	1	2	3	4	5
t_i^n	0	1	1	2	5

Applying the principle mentioned above, we will obtain the following solution:

$$\bar{x}^1=(1, 1, 0, 0, 0), \quad u_1=2,$$

$$\bar{x}^2=(0, 1, 1, 0, 0), \quad u_2=2,$$

$$\bar{x}^3=(0, 0, 1, 1, 0), \quad u_3=4$$

$$\bar{x}^4=(0, 0, 0, 1, 1), \quad u_4=1,$$

$$\bar{x}^5=(0, 0, 0, 0, 1), \quad u_5=1.$$

The time of complex duration $T=\sum_{s=1}^{5} u_s=10$. We formulate the vectors $\bar{y}^1, \bar{y}^2, \bar{y}^3, \bar{y}^4, \bar{y}^5$, by means of the basic vectors:

$$\bar{y}^5=\bar{x}^5, \qquad a_5=1,$$

$$\bar{y}^4=\bar{x}^4-\bar{x}^5, \qquad a_4=0,$$

$$\bar{y}^3=\bar{x}^3-\bar{x}^4+\bar{x}^5, \qquad a_3=1,$$

$$\bar{y}^2=\bar{x}^2-\bar{x}^3+\bar{x}^4-\bar{x}^5, \qquad a_2=0,$$

$$\bar{y}^1=\bar{x}^1-\bar{x}^2+\bar{x}^3-\bar{x}^4+\bar{x}^5, \qquad a_1=1.$$

We obtain a problem of packing

$$x_1+x_3+x_5\to\max$$

$$6x_1+4x_3+7x_5\leqslant 11$$

The problem divides into 3 independent problems for R_1, R_2, R_3:

(i) $x_1+x_3\to\max$,
$6x_1+4x_3\leqslant 11$;

(ii) $x_3\to\max$,
$4x_3\leqslant 11$;

(iii) $x_3+x_5\to\max$,
$4x_3+7x_5\leqslant 11$.

The optimum solution of problem (i): $x_1=x_3=1$; of problem (ii): $x_3=1$; of problem (iii): $x_3=x_5=1$. The solutions of problems (i) and (iii) have the same value $c=2>1$. We introduce to the base, for example the vector

$$\bar{x}^6=(0, 0, 1, 0, 1)=\bar{y}^3+\bar{y}^5=\bar{x}^3-\bar{x}^4+2\bar{x}^5$$

$u_6=\min(u_3/1;\ u_5/2)=1/2$ and we eliminate the vector $\bar{x}^5$. The new solution is:

$$u_1^1=u_1=2;\ u_2^1=u_2=2;\ u_3^1=u_3-\frac{1}{2}=3.5;$$

$$u_4^1=u_4+\frac{1}{2}=1.5;\ u_6^1=0.5;\ T_1=9.5.$$

We substitute $\bar{x}^5=\frac{1}{2}(\bar{x}^6-\bar{x}^3+\bar{x}^4)$ to the formula for $\bar{y}^i$ and we obtain:

$$\bar{y}^1=\bar{x}^1-\bar{x}^2+\frac{1}{2}\bar{x}^3-\frac{1}{2}\bar{x}^4+\frac{1}{2}\bar{x}^6,\quad a_1=\frac{1}{2},$$

$$\bar{y}^2=\bar{x}^2-\frac{1}{2}\bar{x}^3+\frac{1}{2}\bar{x}^4-\frac{1}{2}\bar{x}^6,\quad a_2=\frac{1}{2},$$

$$\bar{y}^3=\frac{1}{2}\bar{x}^3-\frac{1}{2}\bar{x}^4+\frac{1}{2}\bar{x}^6,\quad a_3=\frac{1}{2},$$

$$\bar{y}^4=\frac{1}{2}\bar{x}^1+\frac{1}{2}\bar{x}^3-\frac{1}{2}\bar{x}^6,\quad a_4=\frac{1}{2},$$

$$\bar{y}^5=-\frac{1}{2}\bar{x}^3+\frac{1}{2}\bar{x}^4+\frac{1}{2}\bar{x}^6,\quad a_5=\frac{1}{2}.$$

A new problem of packing

(i) $\frac{1}{2}(x_1+x_2+x_3)\to\max$,
$6x_1+5x_2+4x_3\leqslant 11$;

(ii) $\frac{1}{2}(x_2+x_3+x_4) \to \max,$

$5x_2+4x_3+4x_4 \leqslant 11;$

(iii) $\frac{1}{2}(x_3+x_4+x_5) \to \max,$

$4x_3+4x_4+7x_5 \leqslant 11.$

All the three problems have an optimum value $x=1$. That is the reason why the solution $\{\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4, \bar{x}^6\}$ is the optimum one.

The sequence of the vectors in the solution can be nonunique. Because of that, a problem of determination of such a vector sequence for which the number of operation interruptions will be minimum, can be stated. A groph G in which the nodes correspond to the vectors, will be worked-out. The nodes (i,j) will be connected by means of arcs (i,j) if just one operation of the vector $\bar{x}^i$ precedes at least one operation of the vector $\bar{x}^j$. In the opposite case the nodes will be connected by means of edges. The length of the arc (or edge) will be taken as equal to $l_{ij}=$ $=\sum_{k=1}^{n} |x_k^i - x_k^j|$.

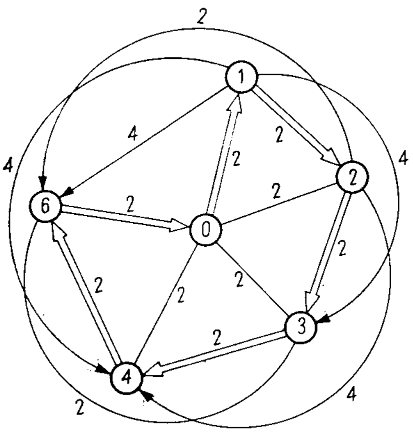

Fig. 2

A fictious node 0 connected with each node i by means of an edge with a length $l_{0i}=\sum_{k=1}^{n} x_k^i$, will be introduced. Let $(0, i_1, i_2, \ldots, i_n, 0)$ be such a Hamiltonian circuit, that, if (i_k, i_s) is an arc, then $k<s$. It can be shown that the Hamiltonian circuit of minimum length determines a sequence of vectors, assuring the minimum number of operation interruptions. At the same time the number of interruptions is equal $\frac{1}{2}L-n$, where L — length of the Hamiltonian circuit. The graph G for the vectors $\bar{x}^1, \bar{x}^2, \bar{x}^3, \bar{x}^4, \bar{x}^6$ is shown in Fig. 2. One of the circuit, which has a minimum length is shown by means of a double line. Its length equals $L=12$. Therefore the vector sequence $\bar{x}^1 \to \bar{x}^2 \to \bar{x}^3 \to \bar{x}^4 \to \bar{x}^6$ determined by this circuit gives a solution with $\frac{1}{2}L-n=1$ interruption during the operations.

6. Optimization when the intervals' lengths are given

Let it be assumed that the lengths Δ_s of all the intervals are given. When such an assumption is made, it is possible to obtain effective algorithms for soluing number of problems. Some of them will be considered.

Let

$$f_i(v_i)=v_i, \quad 0 \leqslant v_i \leqslant \beta_i, \qquad i=1, 2, \ldots, n.$$

All the operations are worked out by means of one kind of resources.

Problem 5 (minimization of resources level). Determine the minimum quantity of resources necessary to work out the complex in a time $T=\sum_{s=1}^{r} \Delta_s$.

We will determine a transportation network with an input x_0, output z, and nodes $i, s, i=1, 2, ..., n, \ s=1, 2, ..., r$. The node x_0 will be connected with each of the nodes i by means of an arc (x_0, i) with a flow capacity w_i; each node i will be connected with each node s by means of an arc (i, s) with flow capacity $\beta_i \Delta_s$. At last, each node s will be connected with the node z by means of an arc (s, z) with a flow capacity $N_0 \Delta_s$, where $N_0=\frac{1}{T}\sum_{i=1}^{n} w_i=\frac{w}{T}$. The maximum flow in this network will be determined applying the Ford-Fulkerson algorithm. If $\varphi_{max}=w$, the flows x_{is} along the arcs (i, s) determine the optimum solution of the problem. If $\varphi_{max}<w$, there exists a set D of nodes s, which were not taken into account at the last step of the algorithm.

Let's compute $M=\sum_{s\in D}\sum_{i\in R_s} x_{is}$ and determine $N_1=\frac{w-M}{T-\sum_{s\in D}\Delta_s}$.

The procedure is repeated for the new value of N_1. After a finite number of steps a flow with the value $\varphi_{max}=w$ is obtained. It determines the optimum solution of the problem $\left(\text{if}\left(\sum_{s\in Q_i}\Delta_s\right)\beta_i\geqslant w_i,\ i=1, 2, ..., n\right)$.

The solution of one more problem for the case $f_i(v_i)=v_i$, $i=1, 2, ..., n$, will be considered.

Problem 6 (problem of an uniform resources utilization). Determine $\{x_{is}\}$, $i\in R_s$, $s=1, 2, ..., r$, which minimize

$$\Phi=\sum_{s=1}^{r}\frac{1}{\Delta_s}\left(\sum_{i\in R_s} x_{is}\right)^2.$$

Let's denote A_k the total volume of the operations, which should be worked-out within the intervals from 1 to k; B_k — the total volume of the operations which can be worked out within the intervals from 1 to k: $T_k=\sum_{s=1}^{k}\Delta_s$, $N(0, r)=w/T$.

Theorem 6. If conditions

$$A_k\leqslant N(0, r)\,T_k\leqslant B_k, \quad k=1, 2, ..., r, \tag{13}$$

are satisfied, then

$$\Phi_{min}=N^2(0, r)\,T=w^2/T.$$

Proof. For the case $r=1$, the theorem is an elementary one. Let's admit that the theorem is true for a network with $(r-1)$ events and we will prove it for r. For that purpose, let's eliminate from the network all the operations which should be worked out within the 1st interval, then the operations which should be worked out within the 2nd one, and can be worked out within the 1st, then the operations which should be worked out within the third interval and can be worked out within the 1st, etc., in such a way that the total volume of eliminated operations equals

$N(0, r)\Delta_1$. After that the whole 1st interval will be eliminated. Conditions (13) are still satisfied for the remaining network with $(r-1)$ intervals. This proves the theorem.

Let k_0 be the nearest interval within which the condition (13) is violated.

If $A_{k_0} > N(0, r)\, T_{k_0}$, problem 6 is separately solved for the network which contains all the operations forming A_{k_0}, and separately for the remaining network.

If $N(0, r)\, T_{k_0} > B_{k_0}$, problem 6 is separately solvod for the network which contains all the operations forming B_{k_0}, and separately for the romaining network.

In such a way the network is divided into sub-networks; for each of them the condition (13) is satisfied.

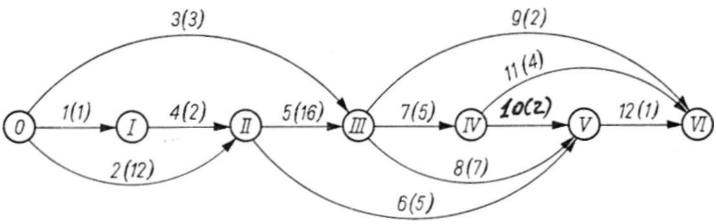

Fig. 3

Example 4. Problem 6 will be solved for the network shown in Fig. 3. (The operations' volumes are given in brackets nearly the corresponding arcs). Let all $\Delta_s = 1$, $s = 1, 2, \ldots, 6$. We have $w = 60$, $T = 6$, $N(0, 6) = 10$. The following is computed:

$$A_1 = 1, \quad B_1 = 16, \quad 1 < 10 < 16,$$

$$A_2 = 15, \quad B_2 = 18, \quad 15 < 20 > 18.$$

The condition (13) has been violated for the second interval. The network will be divided into two parts because $B_2 < T_2 N(0, 6)$. Operations 1, 2, 3, 4 belong to the first sub-network, and the remaining to the second one. When continuing the procedure of checking the condition (13) and dividing the network, four sub-networks are obtained (Fig. 4). For each of the sub-networks G_1, G_2, G_3, G_4 the condition (13) is satisfied. Let's denote by $w(G_i)$ the total volume of the network G_i operations and by $T(G_i)$ the total length of the network G_i intervals.

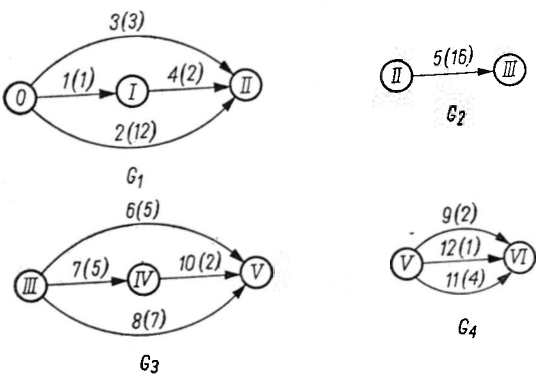

Fig. 4

We have

i	1	2	3	4
$w(G_i)$	18	16	19	7
$T(G_i)$	2	1	2	1

According to the theorem 6

$$\Phi_{\min}=\sum_{i=1}^{4}\frac{w^2(G_i)}{T(G_i)}=\frac{18^2}{2}+\frac{16^2}{1}+\frac{19^2}{2}+\frac{7^2}{1}=647.5.$$

Problem 6 enters as a separate stage to the solution of more complicated problem.

Problem 7. Determine $\{x_{is}\}$ and Δ_s which minimize

$$\sum_{j=1}^{m} c_j \sum_{s=1}^{r} \frac{1}{\Delta_s}\left(\sum_{i\in P_{js}} x_{is}\right)^2, \quad c_j>0, \tag{14}$$

under constraints

$$\sum_{s\in Q_i} x_{is}=w_i, \quad i=1,2,\ldots,n, \tag{15}$$

$$\sum_{s=1}^{r} \Delta_s=T. \tag{16}$$

In [2] an algorithm for solution of problem 7 is given. It consists in consecutive performing of two stages. At the first stage Δ_s which satisfies equation (16) is fixed, and problem 6 is solved. At the second stage the $\{x_{is}\}$ obtained in the previous stage are fixed and the Δ_s which minimizes the following sum is determined:

$$\sum_{s=1}^{r}\frac{B_s}{\Delta_s}, \quad \text{where} \quad B_s=\sum_{j=1}^{m} c_j\left(\sum_{i\in P_{js}} x_{is}\right)^2$$

under constraints (16). The solution of this problem is obvious

$$\Delta_s=\frac{T\sqrt{B_s}}{\sum_{s=1}^{r}\sqrt{B_s}}, \quad s=1,2,\ldots,r.$$

In [2] an iterative algorithm for the solution of problem 6 is proposed, and in [3] the application of a quadratic programming method is given.

7. Synthesis problem of the complex

A complex consisting of n operations is considered. The volume w_i of each operation can have several values. Let's denote by $\sigma_i(w_i)$ a certain function of the operation volume. That function can correspond to the costs connected with carrying out of the operations partly out of a given complex (for example instead of producing all the component parts of a given machine, some of them can be bought ready-made and kept in store).

Problem 8. Determine the volumes $\{w_i\}$ of the operations in such a way that the complex duration time $T_{\min}(\bar{w})$ will be the minimum one under constraint

$$\sum_{i=1}^{n}\sigma_i(w_i)\leqslant\sigma. \tag{17}$$

For the case of ordered events and convex (to bottom) functions $\sigma_i(w_i)$, problem 8 is a problem of convex programming because $T_{\min}(\bar{w})$ is a convex (to bottom) function of $\bar{w}$. For the case $f_i(v_i)=v_i^{1/\alpha}$, $i=1, 2, ..., n$, $\alpha>1$, and resources of one kind, the problem is reduced to a problem of minimizing the equivalent volume of the complex with constraints (17).

Conclusion

A series of problems which seem to be of interest for further development of the presented considerations is given here.

I. All the operations are worked out by means of resources of one kind. It is assumed that all the operations are worked out with a constant intensity, and the resources are a flow in the network.

For the above assumption, estimate the increase of the minimum time of complex duration as a function of the form of $f_i(v_i)$, $i=1, 2, ..., n$, and of the network structure. As it was proved previously, this increase equals 0 for $f_i(v_i)=v_i^{1/\alpha}$, $i=1,2,..., n$, $\alpha>1$, independently of the network structure, and also for arbitrary, convex (to top) functions $f_i(v_i)$ and networks composed of independent operations.

II. In what cases is $T_{\min}(\bar{w})$ a convex (to bottom) function of $\bar{w}$ excepting the case of ordered events?

III. Is the equivalent volume always a convex (to bottom) function of the operations' volumes?

IV. Work-out the algorithm for the solution of network synthes' problems for the case of ordered events.

References

1. Бурков В. Н., Распределение ресурсов как задача оптимального быстродействия. *Автом. и телемех.* 7 (1966).
2. Разумихин Б. С., Задача об оптимальном распределении ресурсов. *Автом. и телемех.* 7 (1965).
3. Воронов А. А., Петрушинин Е. П., Решение задачи оптимального распределения ресурсов методом квадратичного программирования. Москва, 1965.
4. Бурков В. Н.: Оптимальное управление комплексами операций. Докл. на 4 Межд. конгр. ИФАК, Варшава 1969.

Zagadnienia optymalnego rozdziału zasobów

Problemy optymalnego rozdziału ograniczonych zasobów są jednym z najważniejszych kierunków teorii sterowania i planowania sieciowego. Optymalny rozdział zasobów jest w zasadzie zagadnieniem ekstremalnym typu kombinatorycznego. Obecnie nie ma efektywnych i dokładnych metod rozwiązywania takich zagadnień. Wystarczająco opracowana jest jedynie teoria dotycząca zagadnień, w których zakłada się uporządkowanie zdarzeń sieci. W pracy niniejszej rozpatrzono podstawowe wyniki i metody optymalnego rozdziału zasobów uzyskane przy założeniu, że zdarzenia sieci są uporządkowane.

Задачи оптимального распределения ресурсов

Оптимальное распределение ограниченных ресурсов является одним из важнейших направлений в теории сетевого планирования и управления. Задачи оптимального распределения ресурсов являются в основном экстремальными задачами комбинаторного типа. В настоящее время не существует эффективных точных методов их решения в общем случае. Достаточно законченная теория имеется только для задач, в которых предполагается упорядоченность событий сети. В докладе рассматриваются основные результаты и методы оптимального распределения ресурсов при условии заданного упорядочения событий сети.